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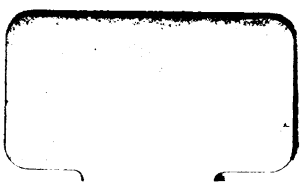

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PLANE AND SOLID

GEOMETRY

TEACHERS' EDITION

BY

GEORGE WENTWORTH

REVISED EDITION

BOSTON, U.S.A.
GINN & COMPANY, PUBLISHERS
The Athenæum Press
1900

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PREFACE.

THIS book is intended for teachers, *and for teachers only*. The publishers will sell it to no persons except teachers of Wentworth's Geometry. Every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn & Company.

It is hoped that young teachers will derive great help from studying the systematic solutions of the exercises, and that every teacher who is pressed for time will find great relief by not being obliged to work out every problem himself.

Criticisms and corrections will be thankfully received.

GEORGE WENTWORTH.

EXETER, N.H., October, 1899.

WENTWORTH'S GEOMETRY.

TEACHERS' EDITION.

BOOK I. PLANE GEOMETRY.

Ex. 1. *Find the complement and the supplement of an angle of* 49°.

$$90° - 49° = 41°.$$
$$180° - 49° = 131°.$$

Ex. 2. *Find the number of degrees in an angle if it is double its complement; if it is one fourth of its complement.*

(1) Let x = the number of degrees in the angle.

Then
$$x + \frac{x}{2} = 90.$$
$$2x + x = 180.$$
$$3x = 180.$$
$$x = 60. \qquad 60°. \ \textit{Ans.}$$

(2) Let x = the number of degrees in the angle.

Then
$$x + 4x = 90.$$
$$5x = 90.$$
$$x = 18. \qquad 18°. \ \textit{Ans.}$$

Ex. 3. *Find the number of degrees in an angle if it is double its supplement; if it is one third of its supplement.*

(1) Let x = the number of degrees in the angle.

Then
$$x + \frac{x}{2} = 180.$$
$$2x + x = 360.$$
$$3x = 360.$$
$$x = 120. \qquad 120°. \ \textit{Ans.}$$

(2) Let x = the number of degrees in the angle.

Then
$$x + 3x = 180.$$
$$4x = 180.$$
$$x = 45. \qquad 45°. \ \textit{Ans.}$$

Ex. 4. *Find the number of degrees in the angle included by the hands of a clock at* 1 *o'clock.* 3 *o'clock.* 5 *o'clock.* 6 *o'clock.*

$30°$; $90°$; $150°$; $180°$. *Ans.*

Ex. 5. *Find the complement and the supplement of an angle that contains* $37° 53' 49''$.

$$90° - 37° 53' 49'' = 52° 6' 11''.$$
$$180° - 37° 53' 49'' = 142° 6' 11''.$$

Ex. 6. *If the complement of an angle is one third of its supplement, how many degrees does the angle contain?*

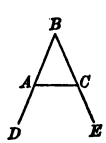
Let x = the number of degrees in the angle.

Then
$$3(90 - x) = 180 - x.$$
$$270 - 3x = 180 - x.$$
$$2x = 90.$$
$$x = 45.$$

$45°$. *Ans.*

Ex. 7. *If the equal sides of an isosceles triangle are produced, the angles on the other side of the base are equal.*

Let the equal sides BA and BC of the isosceles $\triangle ABC$ be produced to D and E, respectively.



To prove that $\angle CAD = \angle ACE$.

Proof.
$\angle BAC = \angle BCA$. § 145

$\angle CAD$ is the supplement of $\angle BAC$,

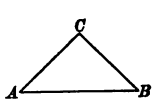
and $\angle ACE$ is the supplement of $\angle BCA$. § 86

$\therefore \angle CAD = \angle ACE$. § 85

Q. E. D.

Ex. 8. *How many degrees are there in each of the acute angles of an isosceles right triangle?*

In the $\triangle ABC$, let C be a rt. $\angle$, and let AC be equal to BC.



To find the value of $\angle A$ and of $\angle B$.

Now $\angle A + \angle B = 90°$. § 135

But $\angle A = \angle B$. § 145

$\therefore \angle A = \angle B = 45°$.

Ex. 9. *If any angle of an isosceles triangle is equal to two thirds of a right angle* (60°), *what is the value of each of the two remaining angles?*

The $\triangle$ is equiangular and each angle is equal to 60°. § 146

Ex. 10. *One angle of a triangle is* 34°. *Find the other angles, if one of them is twice the other.*

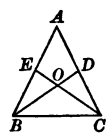
Let x = the number of degrees in the smaller angle.
Then $2x$ = the number of degrees in the larger angle.

$$\therefore x + 2x = 180 - 34.$$
$$3x = 146.$$
$$x = 48\tfrac{2}{3}.$$
$$2x = 97\tfrac{1}{3}. \quad 48° 40'; \ 97° 20'. \ \textit{Ans.}$$

Ex. 11. *If the vertical angle of an isosceles triangle is equal to* 30°, *find the exterior angle included by a side and the base produced.*

Each angle at the base $= \frac{1}{2}(180° - 30°) = 75°$.
Each exterior angle $= 180° - 75° = 105°$.

Ex. 12. *If the vertical angle of an isosceles triangle is equal to* 36°, *find the angle included by the bisectors of the base angles.*



$\angle ABC = \angle ACB = \frac{1}{2}(180° - 36°) = 72°$.
$\angle OBC = \angle OCB = \frac{1}{2}$ of $72° \quad = 36°$.
$\angle BOC = 180° - 2 \times 36° = 180° - 72° = 108°$.

Ex. 13. *The median from the vertex to the base of an isosceles triangle is perpendicular to the base, and bisects the vertical angle.*

In $\triangle ABC$, let AB be equal to AC, and D be the middle point of BC.
To prove that AD is $\perp$ to BC, and $\angle DAB = \angle DAC$.

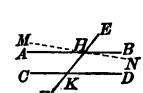
Proof. In the $\triangle BAD$, CAD,

	$AB = AC$, and $BD = CD$,	Hyp.
and	AD is common.	
	$\therefore \triangle BAD = \triangle CAD$.	§ 150
	$\therefore \angle BAD = \angle CAD$,	§ 128
and	$\angle BDA = \angle CDA$.	§ 78
	$\therefore AD$ is $\perp$ to BC.	§ 63

Q. E. D.

Ex. 14. *If two straight lines are cut by a transversal so that the alternate-exterior angles are equal, the two straight lines are parallel.*

Let EF cut the straight lines AB and CD in the points H and K, and let the angles AHE and DKF be equal.



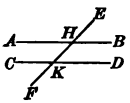
To prove that AB is $\parallel$ to CD.

Proof. Suppose MN drawn through $H \parallel$ to CD.

Then	$\angle MHE = \angle DKF$.	§ 113
But	$\angle AHE = \angle DKF$.	Hyp.
	$\therefore \angle MHE = \angle AHE$.	Ax. 1
	$\therefore MN$ and AB coincide.	§ 60
But	MN is $\parallel$ to CD.	Const.
	$\therefore AB$, which coincides with MN, is $\parallel$ to CD.	Q. E. D.

Ex. 15. *If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.*

Let AB and CD be two parallel lines cut by the transversal EF in the points H and K.



To prove that $\measuredangle EHB$ and DKF are supplementary.

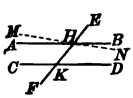
Proof. $\measuredangle CKF$ and DKF are supplementary. § 86

But $\angle CKF = \angle EHB$. § 113

$\therefore \measuredangle EHB$ and DKF are supplementary. Q. E. D.

Ex. 16. *If two straight lines are cut by a transversal so as to make the exterior angles on the same side of the transversal supplementary, the two lines are parallel.*

Let EF cut the straight lines AB and CD in the points H and K, and let the angles EHB and DKF be supplementary.



To prove that AB is $\parallel$ to CD.

Proof. Suppose MN drawn through $H \parallel$ to CD.

Then	$\measuredangle EHN$ and DKF are supplementary.	Ex. 15
But	$\measuredangle EHB$ and DKF are supplementary.	Hyp.
	$\therefore \angle EHN = \angle EHB$.	§ 85
	$\therefore MN$ and AB coincide.	§ 60
But	MN is $\parallel$ to CD.	Const.
	$\therefore AB$, which coincides with MN, is $\parallel$ to CD.	Q. E. D.

Ex. 17. *How many diagonals can be drawn in a polygon of n sides?*

Since a diagonal of a polygon is a line joining two vertices not adjacent (§ 192), from each vertex $(n - 3)$ diagonals can be drawn.

Therefore, from the n vertices, $n(n - 3)$ diagonals can be drawn, but we have thus counted each diagonal twice.

Therefore, in a polygon of n sides, $\frac{1}{2}n(n - 3)$ diagonals can be drawn.

Ex. 18. *How many sides has a polygon if the sum of its interior $\measuredangle$ is twice the sum of its exterior $\measuredangle$? ten times the sum of its exterior $\measuredangle$?*

(1) Let n = the number of sides of the polygon.

Then $(n - 2)2$ = the number of rt. $\measuredangle$ in the sum of the interior $\measuredangle$.

But 4 = the number of rt. $\measuredangle$ in the sum of the exterior $\measuredangle$.

$$\therefore (n - 2)2 = 8.$$
$$2n - 4 = 8.$$
$$2n = 12.$$
$$n = 6.$$

6. *Ans.*

(2) Let n = the number of sides of the polygon.

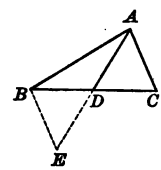
Then $(n-2)\,2 = 40.$

$2n - 4 = 40.$

$2n = 44.$

$n = 22.$ 22. *Ans.*

Ex. 19. *A median of a triangle is less than half the sum of the two adjacent sides.*



Let AD be a median of the $\triangle ABC$.

To prove that $AD < \frac{1}{2}(AB + AC)$.

Proof. Produce AD to E, making DE equal to AD.

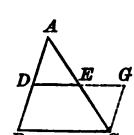
Draw BE.

Now	$\triangle ACD = \triangle EBD.$	§ 143
For	$CD = BD,$	Hyp.
	$AD = ED,$	Const.
and	$\angle ADC = \angle EDB.$	§ 93
	$\therefore AC = BE.$	§ 128
Now	$AB + BE > AE.$	§ 138
That is,	$AE < AB + BE.$	
	$\therefore \frac{1}{2}AE < \frac{1}{2}(AB + BE).$	Ax. 7

Substitute for $\frac{1}{2}AE$ its equal AD, and for BE its equal AC.

Then $AD < \frac{1}{2}(AB + AC).$ Q. E. D.

Ex. 20. *A straight line which bisects two sides of a triangle is parallel to the third side.*



Let the straight line DE bisect the sides AB and AC of the $\triangle ABC$.

To prove that DE is $\parallel$ to BC.

Draw $CG \parallel$ to BA, and produce DE to meet it at G.

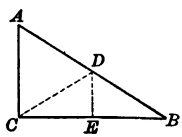
	DE is $\parallel$ to BC if $BCGD$ is a $\square$.	§ 166
	$BCGD$ is a $\square$ if $CG = BD.$	§ 183
	$CG = BD$ if each is equal to $AD.$	Ax. 1
Now	$BD = AD.$	Hyp.
And	$CG = AD$ if $\triangle CGE = \triangle ADE.$	§ 128
But	$\triangle CGE = \triangle ADE.$	§ 139
For	$EC = AE.$	Hyp.
	$\angle GEC = \angle AED.$	§ 93
	$\angle ECG = \angle DAE.$	§ 110
	$\therefore DE$ is $\parallel$ to $BC.$	Q. E. D.

Ex. 21. *The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.*

Let D be the middle point of the hypotenuse AB of the rt. $\triangle ACB$.

To prove that $DA = DB = DC$.

Proof. Draw $DE \perp$ to CB.



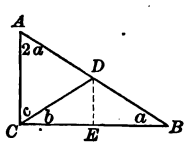
	DE is $\parallel$ to AC.	§ 104
	$\therefore DE$ bisects CB.	§ 188
	$\therefore$ rt. $\triangle DEC$ = rt. $\triangle DEB$.	§ 144
	$\therefore DC = DB$.	§ 128
But	$DB = DA$.	Hyp.
	$\therefore DA = DB = DC$.	Ax. 1

Q. E. D.

Ex. 22. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.*

Let C be the rt. $\angle$ of the rt. $\triangle ACB$, and let $\angle A$ equal $2\angle B$.

To prove that $AB = 2AC$.



Proof. The median $CD = BD = AD$. Ex. 21

Then $\angle b = \angle a$; and $\angle c = \angle 2a$. § 145

Now $a + 2a = 90°$. § 135

$\therefore \angle a = 30°$; $\angle 2a = 60°$; $\angle c = 60°$.

$\therefore \triangle ACD$ is equilateral, § 148

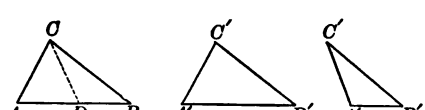
and AD, half of AB, $= AC$. $\therefore AB = 2AC$. Q. E. D.

Ex. 23. *If two triangles have two sides of the one equal, respectively, to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are equal or supplementary, and if equal the triangles are equal.*

Let $AC = A'C'$, $BC = B'C'$, and $\angle B = \angle B'$.

To prove that $\measuredangle A$ and $C'A'B'$ are equal or supplementary.

Proof. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $B'C'$ shall coincide with BC, and $\measuredangle A'$ and A shall be on the same side of BC.



Since $\angle B' = \angle B$, $B'A'$ will fall along BA, and A' will fall at A or at some other point in BA, as D.

If A' falls at A, the $\measuredangle A'B'C'$ and ABC coincide and are equal.

If A' falls at D, the $\measuredangle A'B'C'$ and DBC coincide and are equal.

Since $CD = C'A' = CA$, $\angle A = \angle CDA$. § 145
But $\triangle CDA$ and CDB are supplementary. § 86
$\therefore \triangle A$ and CDB are supplementary. § 85
Q. E. D.

Discussion. The triangles are equal:

1. If the given angles B and B' are both right or both obtuse angles.

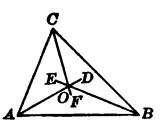
2. If the required angles A and A' are both acute, both right, or both obtuse.

3. If AC and $A'C'$ are not less than BC and $B'C'$, respectively.

Ex. 24. *The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.*

Let the bisectors AD and BE of the $\triangle CAB$ and CBA intersect at O.

To prove that O lies in the bisector CF.



Proof. Now O being in AD is equidistant from AC and AB. § 162

And O being in BE is equidistant from BC and AB. § 162

Hence, O is equidistant from AC and BC.

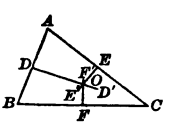
$\therefore O$ is in the bisector CF. § 162

Q. E. D.

Ex. 25. *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.*

Let the $\perp$ bisectors EE' and DD' of the sides AC and AB intersect at O.

To prove that O lies in the $\perp$ bisector FF'.



Proof. Now O being in EE' is equidistant from A and C. § 161

And O being in DD' is equidistant from A and B. § 161

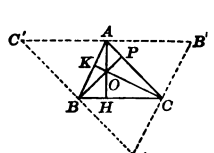
Hence, O is equidistant from B and C.

$\therefore O$ is in the $\perp$ bisector FF'. § 160

Q. E. D.

Ex. 26. *The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.*

Let the $\perp$s be AH, BP, and CK.



To prove that AH, BP, and CK meet in a point.

Proof. Through A, B, C suppose $B'C'$, $A'C'$, $A'B'$, drawn $\parallel$ to CB, CA, BA, respectively.

Then AH is $\perp$ to $B'C'$. § 107

Now $ABCB'$ and $ACBC'$ are $\square$s, § 166

and $AB' = BC$, and $AC' = BC$. § 178

That is, A is the middle point of $B'C'$.

In the same way, B and C are the middle points of $A'C'$ and $A'B'$.

$\therefore AH$, BP, and CK are the $\perp$ bisectors of the sides of the $\triangle A'B'C'$.

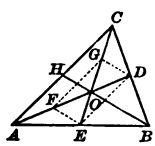
Hence, AH, BP, and CK meet in a point. Ex. 25

Q. E. D.

Ex. 27. *The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the middle of the opposite side.*

Let the two medians AD and CE meet at O.

To prove that the third median BH passes through O.



Proof. Take F the middle point of OA, and G of OC.

Draw GF, FE, ED, and DG.

In $\triangle AOC$, FG is $\parallel$ to AC and equal to $\frac{1}{2}AC$. § 189

ED is $\parallel$ to AC and equal to $\frac{1}{2}AC$. § 189

$\therefore ED$ is $\parallel$ to FG. § 106

Hence, $DGFE$ is a $\square$. § 183

Hence, $AF = FO = OD$, and $CG = GO = OE$. § 184

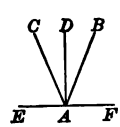
Hence, *any median* cuts off *on any other median* two thirds of the distance from the vertex to the middle of the opposite side.

Therefore, the median from B will cut off AO, two thirds of AD; that is, will pass through O. Q. E. D.

Ex. 28. *If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.*

Let BAC be the given angle, AD the bisector.

Through A draw $EF \perp$ to AD.



To prove that $\angle CAE = \angle BAF$.

Proof. $\angle CAE = 90° - \angle DAC$.

$\angle BAF = 90° - \angle DAB$.

But $\angle DAC = \angle DAB$.

$\therefore \angle CAE = \angle BAF$. Ax. 3

Q. E. D.

Ex. 29. *The bisectors of two supplementary adjacent angles are perpendicular to each other.*

Let $\angle BAC + \angle CAD = 180°$, AE bisect $\angle BAC$, and AF bisect $\angle CAD$.

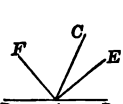
To prove that AF is $\perp$ to AE.

Proof. $2\angle EAC + 2\angle CAF = 180°$.

$\therefore \angle EAC + \angle CAF = 90°$.

$\therefore \angle EAF = 90°$.

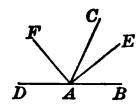
$\therefore AF$ is $\perp$ to AE. Q. E. D.



Ex 30. *If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.*

Let $\measuredangle BAC$, CAD, be two adj. $\measuredangle$, AE bisect $\angle BAC$, AF bisect $\angle CAD$, and $\angle EAF = 90°$.

To prove that $\angle BAC + \angle CAD = 180°$.

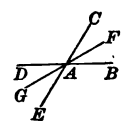


Proof.	$\angle EAC + \angle CAF = 90°$.	Hyp.
	$\therefore 2\angle EAC + 2\angle CAF = 180°$,	
or	$\angle BAC + \angle CAD = 180°$.	Q. E. D.

Ex. 31. *The bisector of one of two vertical angles bisects the other.*

Let $\measuredangle BAC$, DAE, be two vertical $\measuredangle$. Let AF bisect $\angle BAC$. Produce FA to G.

To prove that $\angle DAG = \angle EAG$.



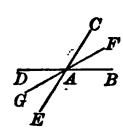
Proof.	$\angle DAG = \angle BAF$,	§ 93
and	$\angle EAG = \angle CAF$.	
But	$\angle BAF = \angle CAF$.	Hyp.
	$\therefore \angle DAG = \angle EAG$.	Ax. 1

Q. E. D.

Ex. 32. *The bisectors of two vertical angles form one line.*

Let $\measuredangle BAC$ and DAE be vertical $\measuredangle$; let AF bisect $\angle BAC$ and AG bisect $\angle DAE$.

To prove that AF and AG form one straight line.



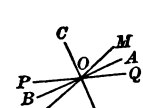
Proof.	$\angle BAC = \angle DAE$.	§ 93
	$\therefore \angle CAF = \angle DAG$.	Ax. 7
	$\therefore \angle CAF + \angle DAG = \angle BAC$.	
Now	$\angle BAC + \angle CAD = 2$ rt. $\measuredangle$.	§ 86
	$\therefore \angle CAF + \angle CAD + \angle DAG = 2$ rt. $\measuredangle$.	
	$\therefore AF$ and AG are in the same straight line.	§ 90

Q. E. D.

Ex. 33. *The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.*

Let MN and PQ intersect at O, and let AB and CD be the bisectors of the $\measuredangle$.

To prove that AB is $\perp$ to CD.

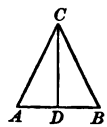


Proof.	$\angle DOQ = \angle MOC$,	Ax. 7
and	$\angle AOQ = \angle AOM$.	Ax. 7
	$\therefore \angle AOD = \angle AOC$.	Ax. 2
But	$\angle AOD + \angle AOC = 180°$.	§ 86
	$\therefore \angle AOD = 90°$.	
	$\therefore AB$ is $\perp$ to CD.	Q. E. D.

Ex. 34. *The bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.*

In $\triangle ABC$, let AC equal BC, and CD bisect $\angle C$.

To prove that $AD = BD$, and CD is $\perp$ to AB.



Proof. The $\triangle ADC$ and BDC are equal. § 143

For $AC = BC$, Hyp.

$\angle ACD = \angle BCD$, Hyp.

and CD is common.

$\therefore AD = BD$, and $\angle ADC = \angle BDC$. § 128

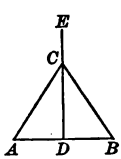
$\therefore$ $\measuredangle ADC$ and BDC are rt. $\measuredangle$. § 63

$\therefore$ CD is $\perp$ to AB. Q. E. D.

Ex. 35. *The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.*

In $\triangle ABC$, let AC equal BC, and let DE be drawn $\perp$ to AB through D, the middle point of AB.

To prove that DE passes through C, and bisects $\angle ACB$.



Proof. DE passes through C, because $AC = BC$. § 160

The $\triangle ACD$ and BCD are equal. § 141

For $AC = BC$, Hyp.

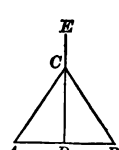
and $\angle CAD = \angle CBD$. § 145

$\therefore \angle ACD = \angle BCD$. § 128

Q. E. D.

Ex. 36. *If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.*

Let DE, the $\perp$ bisector of the base AD of the $\triangle ABC$, pass through C, the vertex of the triangle.



To prove that the $\triangle ABC$ is isosceles.

Proof. Since C lies in the $\perp$ bisector DE,

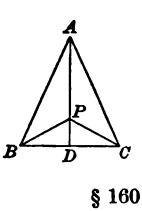
$AC = BC$. § 160

$\therefore \triangle ABC$ is isosceles. § 120

Q. E. D.

Ex. 37. *Any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.*

Let AD be the bisector of the vertical $\angle BAC$ of the isosceles $\triangle ABC$. Let P be any point in AD.



To prove that $PB = PC$.

Proof. The bisector of the vertical $\angle$ of an isosceles $\triangle$ is the $\perp$ bisector of the base. Ex. 34

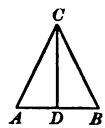
Hence, any point P in this $\perp$ bisector is equidistant from the extremities of the base. § 160

Q. E. D.

Ex. 38. *If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.*

Let CD, the bisector of the $\angle C$ of the $\triangle ABC$, be $\perp$ to AB.

To prove that $\triangle ABC$ is isosceles.



Proof. In the rt. $\triangle ADB$ and ADC,

CD is common,

and $\angle ACD = \angle BCD$. Hyp.

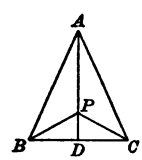
$\therefore \triangle ADB = \triangle ADC$. § 142

$\therefore AC = BC$. § 128

$\therefore \triangle ABC$ is isosceles. Q. E. D.

Ex. 39. *If two isosceles triangles are on the same base, a straight line passing through their vertices is perpendicular to the base and bisects the base.*

Let ABC and PBC be two isosceles $\triangle$ on the same base BC. Let AP produced meet BC at D.



To prove that D is the middle point of BC.

Proof. Since A is equidistant from B and C, and P is also equidistant from B and C, the line APD is $\perp$ to BC at its middle point. § 161

$\therefore D$ is the middle point of BC.

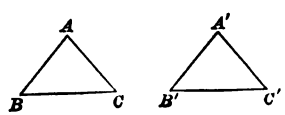
Q. E. D.

Ex. 40. *Two isosceles triangles are equal when a side and an angle of the one are equal, respectively, to the homologous side and angle of the other.*

Let ABC and $A'B'C'$ be two isosceles $\triangle$ having a side and an angle of the one equal to the homologous side and the homologous angle of the other.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. Since an angle of the one is equal to the homologous angle of the other, and in each $\triangle$ two $\measuredangle$ are equal, the three $\measuredangle$ of one $\triangle$ are equal, respectively, to the three $\measuredangle$ of the other.



$\therefore \triangle ABC = \triangle A'B'C'$. § 139

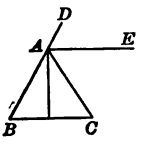
Q. E. D.

Ex. 41. *The bisector of an exterior angle of an isosceles triangle, formed by producing one of the legs through the vertex, is parallel to the base.*

In $\triangle ABC$ let AB be equal to AC, and let AE bisect the $\angle CAD$ formed by producing BA to D.

To prove that AE is $\parallel$ to BC.

Proof. $\angle CAD = \angle ABC + \angle ACB$. § 137
But $\angle CAD = 2\angle DAE$, Hyp.
and $\angle ABC = \angle ACB$. § 145
$\therefore 2\angle DAE = 2\angle ABC$.
$\therefore \angle DAE = \angle ABC$. Ax. 7
$\therefore AE$ is ∥ to BC. § 114



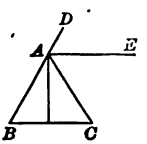
Q. E. D.

Ex. 42. *If the bisector of an exterior angle of a triangle is parallel to one side, the triangle is isosceles.*

In $\triangle ABC$, let AE bisect $\angle CAD$ formed by producing BA to any point D, and let AE be ∥ to BC.

To prove that $\triangle ABC$ is isosceles.

Proof. $\angle DAE = \angle ABC$, § 112
and $\angle CAE = \angle ACB$. § 110
Now $\angle DAE = \angle CAE$. Hyp.
$\therefore \angle ABC = \angle ACB$. Ax. 1
$\therefore AB = AC$. § 147
$\therefore \triangle ABC$ is isosceles.



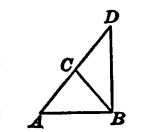
Q. E. D.

Ex. 43. *If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.*

In $\triangle ABC$, let AC be equal to BC. Produce AC, making CD equal AC, and draw DB.

To prove that BD is ⊥ to AB.

Proof. In $\triangle BCD$, $CD = CB$ (since each $= AC$).
$\therefore \angle CBD = \angle CDB$; § 145
also $\angle CBA = \angle CAB$.
$\therefore$ by addition, $\angle ABD = \angle CDB + \angle CAB$. Ax. 2
But the sum of these $\measuredangle s = 180°$. § 129
$\therefore \angle ABD = 90°$.
$\therefore BD$ is ⊥ to AB.



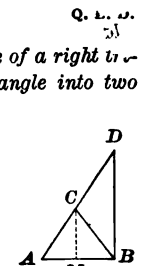
Q. E. D.

Ex. 44. *A line drawn from the vertex of the right angle of a right triangle to the middle point of the hypotenuse divides the triangle into two isosceles triangles.*

Let ABD be a rt. $\triangle$, with rt. $\angle$ at B. Draw BC from B to the middle point of the hypotenuse AD.

To prove that $\measuredangle ACB$ and BCD are isosceles.

Proof. From C draw $CM \perp$ to AB.
Then CM is ∥ to DB, § 104
CM bisects AB. § 188



In rt. $\triangle AMC$ and BMC, $AM = BM$, and CM is common.

$\therefore \triangle AMC = \triangle BMC.$ § 144

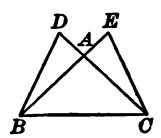
$\therefore AC = BC.$ § 128

$\therefore AC = BC = CD.$

$\therefore \triangle ABC$ and BCD are isosceles. Q. E. D.

Ex. 45. *If the equal sides of an isosceles triangle are produced through the vertex so that the external segments are equal, the extremities of these segments will be equally distant from the extremities of the base, respectively.*

In the $\triangle ABC$, let BA be equal to CA. Let BA and CA be produced to E and D, respectively, so that $AE = AD$.



To prove that $DB = EC$.

Proof. The $\triangle BAD$ and CAE are equal. § 143

For $BA = CA$, Hyp.

$AD = AE$, Hyp.

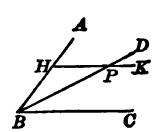
and $\angle BAD = \angle CAE$. § 93

$\therefore DB = EC$. § 128

Q. E. D.

Ex. 46. *If through any point in the bisector of an angle a line is drawn parallel to either of the sides of the angle, the triangle thus formed is isosceles.*

Let BD bisect $\angle ABC$. Through P, any point of BD, draw $HK \parallel$ to BC.



To prove that $\triangle BHP$ is isosceles.

Proof. $\angle PBC = \angle PBA$. Hyp.

$\angle PBC = \angle HPB$. § 110

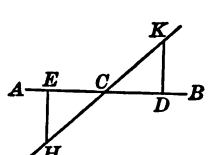
$\therefore \angle PBA = \angle HPB$. Ax. 1

$\therefore BH = PH$. § 147

$\therefore \triangle BHP$ is isosceles. Q. E. D.

Ex. 47. *Through any point C in the line AB an intersecting line is drawn, and from any two points in this line equidistant from C perpendiculars are dropped on AB or AB produced. Prove that these perpendiculars are equal.*

Let HK be a line through the point C in the line AB. Let CH be equal to CK, and let HE and KD be $\perp$ to AB.



To prove that $HE = KD$.

The rt. $\triangle ECH$ and DCK are equal. § 141

For $CH = CK$, Hyp.

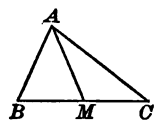
and $\angle ECH = \angle DCK$. § 93

$\therefore HE = KD$. § 128

Q. E. D.

Ex. 48. *If the median drawn from the vertex of a triangle to the base is equal to half the base, the vertical angle is a right angle.*

Let AM be the median of the $\triangle BCA$, and let AM be equal to $\frac{1}{2}\,BC$.



To prove that $\angle BAC$ is a rt. $\angle$.

Proof. In $\triangle BAM$, $AM = BM$. Hyp.

$\therefore \angle MBA = \angle MAB$. § 145

In $\triangle MAC$, $AM = CM$. Hyp.

$\therefore \angle MCA = \angle MAC$. § 145

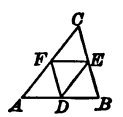
$\therefore \angle MBA + \angle MCA = \angle MAB + \angle MAC$, Ax. 2

or $\angle MBA + \angle MCA = \angle BAC$. $\therefore \angle BAC$ is a rt. $\angle$. § 129

Q. E. D.

Ex. 49. *The lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.*

In $\triangle ABC$, let D, E, F be the middle points of the sides AB, BC, CA, respectively.



To prove that $\triangle ADF$, FED, DBE, FEC are equal.

Proof. The $\triangle ADF$ and FED are equal. § 150

For DF is common,

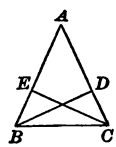
$AD = FE$, and $AF = DE$. § 189

Similarly, $\triangle DBE = \triangle FED = \triangle FEC$.

Q. E. D.

Ex. 50. *The altitudes upon the legs of an isosceles triangle are equal.*

In $\triangle ABC$, let AB be equal to AC, and let BD be $\perp$ to AC, $CE \perp$ to AB.



To prove that $BD = CE$.

Proof. The rt. $\triangle BCE$ and BCD are equal. § 141

For BC is common,

and $\angle CBE = \angle BCD$. § 145

$\therefore BD = CE$. § 128

Q. E. D.

Ex. 51. *If the altitudes upon two sides of a triangle are equal, the triangle is isosceles.*

In $\triangle ABC$, let BD be $\perp$ to AC, $CE \perp$ to AB, and BD equal to CE.

To prove that $\triangle ABC$ is isosceles.

Proof. The rt. $\triangle BCD$ and BCE are equal. § 142

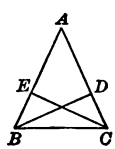
For BC is common,

and $BD = CE$. Hyp.

$\therefore \angle ACB = \angle ABC$. § 128

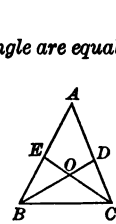
$\therefore AB = AC$. § 147

$\therefore \triangle ABC$ is isosceles. Q. E. D.



Ex. 52. *The medians drawn to the legs of an isosceles triangle are equal.*

In $\triangle ABC$, let AB be equal to AC, and BD and CE be the medians to the equal sides.



To prove that $BD = CE$.

Proof. The $\triangle BCD$ and BCE are equal. § 143

For BC is common,

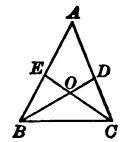
$CD = BE$, Ax. 7

and $\angle ACB = \angle ABC$. § 145

$\therefore BD = CE$. § 128 Q. E. D.

Ex. 53. *If the medians to two sides of a triangle are equal, the triangle is isosceles.*

In $\triangle ABC$, let BD, CE be two medians meeting at O, and BD be equal to CE.



To prove that $\triangle ABC$ is isosceles.

Proof. The $\triangle BOE$ and COD are equal. § 143

For $BO = CO$, and $OE = OD$, Exs. 27, 52

and $\angle BOE = \angle COD$. § 93

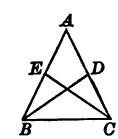
$\therefore BE = CD$. § 128

$\therefore AB = AC$. Ax. 6

Q. E. D.

Ex. 54. *The bisectors of the base angles of an isosceles triangle are equal.*

In $\triangle ABC$, let AB be equal to AC, and let BD bisect the $\angle ABC$, and CE the $\angle ACB$.



To prove that $BD = CE$.

Proof. The $\triangle CDB$ and BEC are equal. § 139

For BC is common,

$\angle BCD = \angle CBE$, § 145

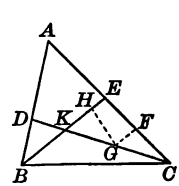
and $\angle CBD = \angle BCE$. Ax. 7

$\therefore BD = CE$. § 128

Q. E. D.

Ex. 55. Opposite Theorem. *If a triangle is not isosceles, the bisectors of the base angles are not equal.*

In the $\triangle ABC$, let $\angle ABC$ be greater than $\angle ACB$; let BE bisect $\angle ABC$, and CD bisect $\angle ACB$.



To prove that $CD > BE$.

Proof. Now $KC > KB$. § 153

Now $CD > BE$, if $KD >$ or $= KE$.

But suppose $KD < KE$. Lay off KH equal to KD and KG equal to KB, draw HG, and draw $GF \parallel$ to BE.

$\triangle KDB = \triangle KHG$. § 143

For $KD = KH$, and $KB = KG$, Const.

and $\angle DKB = \angle HKG.$ § 93

$\therefore \angle KHG = \angle KDB.$ § 128

$\therefore \angle KEC$ is greater than $\angle KHG.$

$\therefore GF > HE.$

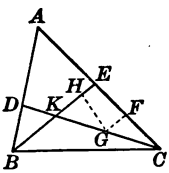
But $\angle GFC$ is greater than $\angle FCG$ ($\frac{1}{2}\angle ACB$).

$\therefore CG > GF$, and $> HE.$

$\therefore KC - KG > KE - KH,$

or $KC + KD > KB + KE,$

or $CD > BE.$ Q. E. D.



Ex. 56. *State the converse theorem of Ex. 54. Is the converse theorem true?*

Converse Theorem. If the bisectors of two angles of a triangle are equal, the triangle is isosceles. (For proof, see page 219.)

Ex. 57. *The perpendiculars dropped from the middle point of the base upon the legs of an isosceles triangle are equal.*

In $\triangle ABC$, let AB be equal to AC, D be the middle point of BC, $DE \perp$ to AB, $DF \perp$ to AC.

To prove that $DE = DF.$

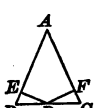
Proof. The rt. $\triangle BDE$ and CDF are equal. § 142

For $BD = CD,$ Hyp.

and $\angle EBD = \angle FCD.$ § 145

$\therefore DE = DF.$ § 128

Q. E. D.



Ex. 58. *State and prove the converse.*

If the perpendiculars dropped from the middle point of one side of a triangle to the other two sides are equal, the triangle is isosceles.

Let ABC be any $\triangle$, D the middle point of BC, $DE \perp$ to AB, $DF \perp$ to AC, and DE equal to DF.

To prove that $\triangle ABC$ is isosceles.

Proof. The rt. $\triangle BDE$ and CDF are equal. § 151

For $BD = DC,$ and $DE = DF.$ Hyp.

$\therefore \angle ABC = \angle ACB.$ § 128

$\therefore AB = AC.$ § 147

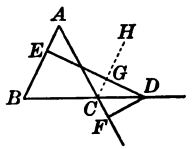
$\therefore \triangle ABC$ is isosceles. Q. E. D.

Ex. 59. *The difference of the distances from any point in the base produced of an isosceles triangle to the equal sides of the triangle is constant.*

Let ABC be an isosceles $\triangle$, and let DE and DF be $\perp$s drawn from any point D of the base produced. Also let DE be greater than DF.

To prove that $DE - DF$ is constant.

Proof. Draw $CH \parallel$ to BA.



Now $\angle ABC = \angle ACB$, § 145

$\angle ABC = \angle HCD$, § 112

and $\angle DCF = \angle ACB$. § 93

$\therefore \angle DCG = \angle DCF$. Ax. 1

The rt. $\triangle DGC$ and DFC are equal. § 141

For DC is common, and $\angle DCG = \angle DCF$.

$\therefore DG = DF$. § 128

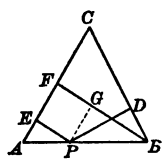
$\therefore DE - DF = DE - DG = GE$, the $\perp$ distance between BA and CH.

$\therefore DE - DF$ is constant. Q. E. D.

Ex. 60. *The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the legs is constant, and equal to the altitude upon one of the legs.*

Let ABC be an isosceles $\triangle$, AB the base, P any point in AB, $PD \perp$ to BC, $PE \perp$ to AC, and draw $BF \perp$ to the leg AC.

To prove that $PD + PE = BF$.



Proof. Draw $PG \perp$ to BF.

PE is $\parallel$ to BF, and PG is $\parallel$ to AC. § 104

$\therefore \angle BAC = \angle BPG$; § 112

also $\angle ABC = \angle BAC$. § 145

$\therefore \angle ABC = \angle BPG$. Ax. 1

The rt. $\triangle PBD$ and BPG are equal. § 141

For PB is common, and $\angle ABC = \angle BPG$.

$\therefore PD = BG$. § 128

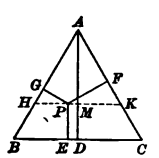
Now $PE = GF$. § 180

$\therefore PD + PE = BG + GF = BF$. Q. E. D.

Ex. 61. *The sum of the perpendiculars dropped from any point within an equilateral triangle to the three sides is constant, and equal to the altitude.*

Let ABC be an equilateral $\triangle$, BC the base, AD the altitude, P any point within the $\triangle$, PE, PF, PG is dropped from P to the sides BC, AC, AB, respectively.

To prove that $PE + PF + PG = AD$.



Proof. Through P draw $HK \parallel$ to BC, meeting AD at M. Now the three altitudes of an equilateral triangle are equal. Ex. 50

$\angle AHK = \angle ABC = 60°$. § 112

$\angle AKH = \angle ACB = 60°$.

$\therefore \triangle AHK$ is equiangular, § 131

and, therefore, equilateral. § 148

$\therefore PF + PG = AM$. Ex. 60

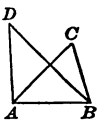
Now $PE = MD.$ §§ 104, 180

$\therefore PE + AM = MD + AM = AD.$ Ax. 2

That is, $PE + PF + PG = AD.$ Q. E. D.

Ex. 62. *ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equals AD, show that BC cannot equal BD.*

To prove that BC and BD are unequal.



Proof. Since AC and AD do not coincide, one of these lines must make with AB an angle greater than the other.

If $\angle DAB$ is greater than $\angle CAB$, then BD is greater than BC. § 154

If $\angle CAB$ is greater than $\angle DAB$, then BC is greater than BD. § 154

That is, BD and BC are unequal. Q. E. D.

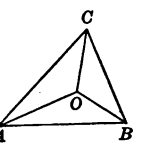
Ex. 63. *The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.*

Let O be a point within the $\triangle ABC$.

To prove that

$$OA + OB + OC < AB + BC + CA,$$

and $$OA + OB + OC > \tfrac{1}{2}(AB + BC + CA).$$



Proof. $OA + OB < AC + BC$ (§ 100), but $> AB$; § 49

$OB + OC < AB + AC$, but $> BC$;

$OC + OA < AB + BC$, but $> AC$.

$\therefore 2(OA + OB + OC) < 2(AB + BC + AC)$, but $> AB + BC + AC$, Ax. 4

or $OA + OB + OB < AB + BC + AC$, but $> \tfrac{1}{2}(AB + BC + AC)$. Ax. 7

Q. E. D.

Ex. 64. *If from any point in the base of an isosceles triangle parallels to the legs are drawn, a parallelogram is formed whose perimeter is constant, being equal to the sum of the legs of the triangle.*

In $\triangle ABC$, let AB be equal to AC, let D be any point in BC, DE be $\parallel$ to CA, $DF \parallel$ to BA.

To prove that $AEDF$ is a $\square$, and its perimeter is equal to $AB + AC$.

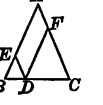
Proof. $AEDF$ is a $\square$ by definition. § 166

Now $\angle EDB = \angle C$, § 112

and $\angle B = \angle C$. § 145

$\therefore \angle EDB = \angle B$. Ax. 1

$\therefore ED = EB$. § 147



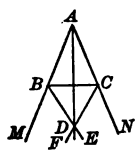
Similarly, $FD = FC$.

$\therefore AE + ED + DF + AF = AB + AC.$ Q. E. D.

Ex. 65. *The bisector of the vertical angle A of a triangle ABC, and the bisectors of the exterior angles at the base formed by producing the sides AB and AC, meet in a point which is equidistant from the base and the sides produced.*

Let ABC be a $\triangle$. Produce AB to any point M, AC to any point N. Let the bisectors BE, CF of the $\measuredangle MBC$, NCB meet at the point D.

To prove that bisector of $\angle A$ passes through D, and D is equidistant from AM, AN, and BC.



Proof. Every point in BE is equidistant from BM and BC. § 162

And every point in CF is equidistant from CN and BC. § 162

$\therefore$ the point D, which is common to BE and CF, is equidistant from BM, BC, and CN.

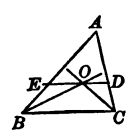
$\therefore$ the bisector of the $\angle A$ passes through D. § 162

Q. E. D.

Ex. 66. *If the bisectors of the base angles of a triangle are drawn, and through their point of intersection a line is drawn parallel to the base, the length of this parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.*

Let ED be drawn $\parallel$ to the base BC of the $\triangle ABC$ through O, the intersection of the bisectors OB, OC of the $\measuredangle ABC$, ACB.

To prove that $ED = EB + DC$.



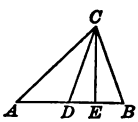
Proof.	$\angle EBO = \angle OBC$,	Hyp.
	$\angle EOB = \angle OBC$.	§ 110
	$\therefore \angle EBO = \angle EOB$.	Ax. 1
	$\therefore EO = EB$.	§ 147
Similarly,	$OD = DC$.	
	$\therefore EO + OD = EB + DC$.	Ax. 2
That is,	$ED = EB + DC$.	

Q. E. D.

Ex. 67. *The bisector of the vertical angle of a triangle makes with the perpendicular from the vertex to the base an angle equal to half the difference of the base angles.*

In the $\triangle ABC$, let CD bisect $\angle C$, and CE be $\perp$ to AB, and let $\angle B$ be greater than $\angle A$.

To prove that $\angle DCE = \frac{1}{2}(\angle B - \angle A)$.



Proof. $\angle DCE = \angle ACE - \angle ACD$.

Now $\angle ACE = 90° - \angle A$, § 135

and $\angle ACD = \frac{1}{2}\angle ACB$. Hyp.

$\therefore \angle DCE = 90° - \angle A - \frac{1}{2}\angle ACB$.

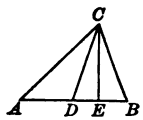
Also $\angle ACB = 180° - \angle A - \angle B$. § 131

$\therefore \frac{1}{2}\angle ACB = 90° - \frac{1}{2}\angle A - \frac{1}{2}\angle B$. Ax. 7

$\therefore \angle DCE = 90° - \angle A - (90° - \frac{1}{2}\angle A - \frac{1}{2}\angle B)$

$= 90° - \angle A - 90° + \frac{1}{2}\angle A + \frac{1}{2}\angle B$

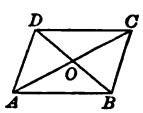
$= \frac{1}{2}\angle B - \frac{1}{2}\angle A$. Q. E. D.



Ex. 68. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

Let the diagonals AC, BD of the quadrilateral $ABCD$ meet at O, and let AO be equal to CO, BO to DO.

To prove that figure $ABCD$ is a parallelogram.



Proof. The △ ABO and DCO are equal. § 143

For $AO = CO$, Hyp.

$BO = DO$, Hyp.

and $\angle AOB = \angle COD$. § 93

$\therefore AB = DC$, and $\angle BAO = \angle DCO$. § 128

$\therefore AB$ is ∥ to DC. § 111

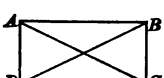
$\therefore$ figure $ABCD$ is a parallelogram. § 183

Q. E. D.

Ex. 69. *The diagonals of a rectangle are equal.*

Let $ABCD$ be a rectangle.

To prove that $AC = BD$.



Proof. The rt. △ ABD and DCA are equal. § 144

For AD is common,

and $AB = DC$. § 178

$\therefore AC = BD$. § 128

Q. E. D.

Ex. 70. *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

Let $ABCD$ be a parallelogram, and let AC be equal to BD.

To prove that $ABCD$ is a rectangle.

Proof. The △ ABD and DCA are equal. § 150

For AD is common,

$AC = BD$, Hyp.

and $AB = DC$. § 178

$\therefore \angle BAD = \angle ADC$. § 128

Now $\angle BAD + \angle ADC = 180°$. § 115

$\therefore \angle BAD = \angle ADC = 90°$.

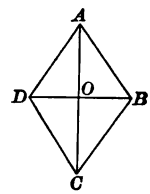
Similarly, $\angle ABC = \angle DCB = 90°$

Therefore, $ABCD$ is a rectangle. Q. E. D.

Ex. 71. *The diagonals of a rhombus are perpendicular to each other, and bisect the angles of the rhombus.*

Let $ABCD$ be a rhombus, and let AC meet BD at O.

To prove that AC is $\perp$ to BD, and that $\angle BAO = \angle DAO$, etc.



Proof. The $\triangle OAB$ and OAD are equal. § 150

For OA is common,

$AB = AD$, § 170

and $OB = OD$. § 184

$\therefore \angle BAO = \angle DAO$. § 128

Also $\angle AOB = \angle AOD$.

$\therefore \angle AOB = 90°$. § 63

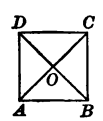
$\therefore AC$ is $\perp$ to BD.

Similarly, we may prove that $\angle ABO = \angle CBO$, etc. Q. E. D.

Ex. 72. *The diagonals of a square are perpendicular to each other, and bisect the angles of the square.*

Let $ABCD$ be a square, and AC meet BD at O.

To prove that AC is $\perp$ to BD, and that $\angle BAO = \angle DAO$, etc.



Proof. The $\triangle OAB$ and OAD are equal. § 150

For OA is common,

$AB = AD$, § 168

and $OB = OD$. § 184

$\therefore \angle BAO = \angle DAO$, § 128

and $\angle AOB = \angle AOD$.

$\therefore \angle AOB = 90°$. § 63

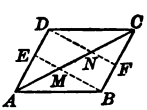
$\therefore AC$ is $\perp$ to BD.

Similarly, we may prove that $\angle ABO = \angle CBO$, etc. Q. E. D.

Ex. 73. *Lines from two opposite vertices of a parallelogram to the middle points of the opposite sides trisect the diagonal.*

In the $\square ABCD$, let E and F be the middle points of the opposite sides AD and BC. Let BE cut AC at M, and FD cut AC at N.

To prove that $AM = MN = NC$.



Proof. In the figure $EBFD$,

ED is $\parallel$ to BF. § 166

Also, since $AD = BC$, § 178

$ED = BF$. Ax. 7

$\therefore DF$ is $\parallel$ to EB. § 183

In the $\triangle ADN$, $AM = MN$. § 188

In the $\triangle CMB$, $NC = MN$. § 188

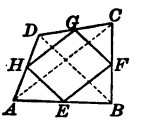
$\therefore AM = MN = NC$. Ax. 1

Q. E. D.

Ex. 74. *The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram.*

Let $ABCD$ be a quadrilateral, E, F, G, H the middle points of the sides, taken in order.

To prove that the figure $EFGH$ is a $\square$.



Proof. Draw the diagonals AC, BD.

EF and HG are $\parallel$ to AC, § 189

and $EF = HG = \frac{1}{2}AC$. § 189

$\therefore EF$ is $\parallel$ to HG. § 106

$\therefore$ figure $EFGH$ is a $\square$. § 183

Q. E. D.

Ex. 75. *The lines joining the middle points of the sides of a rhombus, taken in order, enclose a rectangle.*

Let $ABCD$ be a rhombus, E, F, G, H the middle points of its sides, taken in order.

To prove that the figure $EFGH$ is a rectangle.

Proof. Draw the diagonals AC, BD, meeting at O.

EF is $\parallel$ to AC and equal to $\frac{1}{2}AC$, § 189

and HG is $\parallel$ to AC and equal to $\frac{1}{2}AC$.

$\therefore EF$ is $\parallel$ to HG, § 106

and $EF = HG$. Ax. 1

$\therefore$ figure $EFGH$ is a $\square$. § 183

The $\measuredangle$ formed at O are rt. $\measuredangle$. Ex. 71

$\therefore$ the $\measuredangle$ of the $\square$ $EFGH$ are rt. $\measuredangle$. § 176

Also BD is not equal to AC.

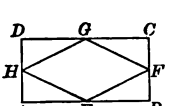
$\therefore EF$ is not equal to FG.

$\therefore$ the figure $EFGH$ must be a rectangle. § 167

Q. E. D.

Ex. 76. *The lines joining the middle points of the sides of a rectangle (not a square), taken in order, enclose a rhombus.*

Let $ABCD$ be a rectangle (not a square), and E, F, G, H be the middle points of its sides, taken in order.



To prove that the figure $EFGH$ is a rhombus.

Proof. The rt. $\triangle AEH$ and BEF are equal. § 144

For $AE = BE$, Hyp.

and $AH = BF$. § 178, Ax. 7

$\therefore EH = EF$. § 128

Similarly, $EH = HG = GF$.

$\therefore EFGH$ is a square or a rhombus. §§ 169, 170

Now the figure $EFGH$ cannot be a square.

For if it were a square, then $\angle HEF$ would be equal to 90°,

and $\angle AEH + \angle BEF$ would be equal to 90°. § 88

But $\angle AEH = \angle BEF$. § 128

$\therefore \angle AEH$ would be equal to 45°.

$\therefore \angle AHE$ would be equal to 45°. § 135

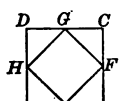
$\therefore AE$ would be equal to AH. § 147

But this conclusion is contrary to the hypothesis that the rectangle $ABCD$ is not a square.

$\therefore$ the figure $EFGH$ must be a rhombus. Q. E. D.

Ex. 77. *The lines joining the middle points of the sides of a square, taken in order, enclose a square.*

Let $ABCD$ be a square, and E, F, G, H the middle points of its sides, taken in order.



To prove that the figure $EFGH$ is a square.

Proof. The rt. $\triangle AEH$, BEF, CFG, DGH are isosceles because their legs are each half of a side of the square.

And these four $\triangle$ are also equal. § 144

$\therefore HE = EF = FG = GH$.

Also in each $\triangle$ the $\angle$ at the base are equal. § 145

$\therefore$ each of the base $\angle = 45°$. § 135

$\therefore \angle FEH = 180° - \angle AEH - \angle BEF = 90°$. § 89

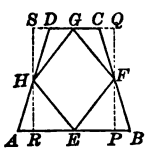
Similarly, the other $\angle$ of $EFGH$ are rt. $\angle$.

$\therefore$ the figure $EFGH$ is a square. § 168

Q. E. D.

Ex. 78. *The lines joining the middle points of the sides of an isosceles trapezoid, taken in order, enclose a rhombus or a square.*

Let $ABCD$ be an isosceles trapezoid, and let EF, FG, GH, HE join the middle points of its sides, taken in order.



To prove that the figure $EFGH$ is a rhombus or a square.

Proof. Through the middle points F, H of the legs draw PQ, $RS \perp$ to the bases, PQ meeting AB at P and DC produced at Q, RS meeting AB at R and CD produced at S. Then PQ is $\parallel$ to RS. § 104

Also RP is $\parallel$ to SQ. Hyp.

$\therefore$ figure $PQSR$ is a $\square$. § 166

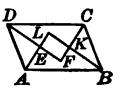
And, by construction, $PQSR$ is a rectangle or a square.

$\therefore$ the figure $EFGH$ is a rhombus (Ex. 76) or a square (Ex. 77).

Q. E. D.

Ex. 79. *The bisectors of the angles of a rhomboid enclose a rectangle.*

Let the quadrilateral $EFKL$ be formed by the bisectors of the angles of the rhomboid $ABCD$.



To prove that $EFKL$ is a rectangle.

Proof. Since DC and AB are $\parallel$, § 166

$\angle CDA + \angle BAD = 180°$. § 115

$\therefore \angle EDA + \angle EAD = 90°$. Ax. 7

$\therefore \angle DEA$ is a rt. $\angle$. § 131

$\therefore \angle FEL$ is a rt. $\angle$. § 93

In like manner, the $\measuredangle ELK$, LKF, KFE are rt. $\measuredangle$.

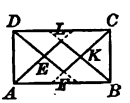
$\therefore EFKL$ is a rectangle. § 167

Q. E. D.

Ex. 80. *The bisectors of the angles of a rectangle enclose a square.*

Let the quadrilateral $EFKL$ be formed by the bisectors of the $\measuredangle$ of the rectangle $ABCD$.

To prove that $EFKL$ is a square.



Proof. The quadrilateral $EFKL$ is a rectangle by the same proof as in Ex. 79.

The rt. $\measuredangle EDA$ and KBC are equal. § 141

For $DA = CB$, § 178

and $\angle EDA = 45° = \angle KBC$. Hyp.

$\therefore ED = EA = KC = KB$. § 128

In like manner, $\measuredangle FDC$ and LAB are equal,

and $FD = FC = LA = LB$.

$\therefore FD - ED = FC - KC = LB - KB = LA - EA$. Ax. 3

That is, $EF = FK = LK = EL$.

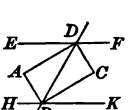
$\therefore EFKL$ is a square. § 168

Q. E. D.

Ex. 81. *If two parallel lines are cut by a transversal, the bisectors of the interior angles form a rectangle.*

Let the quadrilateral $ABCD$ be formed by the bisectors of the interior $\measuredangle$ of the $\parallel$ lines cut by the transversal BD.

To prove that $ABCD$ is a rectangle.



Proof. $\measuredangle ADC$ and ABC are rt. $\measuredangle$. Ex. 29

Now $\angle EDB + \angle DBH = 2$ rt. $\measuredangle$. § 115

$\therefore \angle ADB + \angle ABD = 1$ rt. $\angle$. Ax. 7

$\therefore \angle DAB$ is a rt. $\angle$. § 131

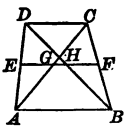
In like manner, $\angle DCB$ is a rt. $\angle$.

$\therefore ABCD$ is a rectangle. § 167

Q. E. D.

Ex. 82. *The median of a trapezoid passes through the middle points of the two diagonals.*

Let $ABCD$ be a trapezoid, AB, DC the $\parallel$ sides, AC, BD the diagonals, EF the median.



To prove that EF bisects AC and BD.

Proof. Let G, H be the middle points of AC, BD.

EF is $\parallel$ to AB and DC. § 190

E is the middle point of AD. Hyp.

$\therefore EF$ passes through H. § 188

For a similar reason EF passes through G.

$\therefore EF$ bisects the two diagonals. Q. E. D.

Ex. 83. *The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.*

Let $ABCD$ be a trapezoid, AB, DC the $\parallel$ sides, AC, BD its diagonals, E, F their middle points.

To prove that $EF = \frac{1}{2}(AB - DC)$.

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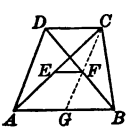
Proof. Draw CF, and produce it to meet AB at G.



The $\triangle BFG$ and CFD are equal. § 139

For $BF = DF$, Hyp.

$\angle BFG = \angle CFD$, § 93

and $\angle GBF = \angle CDF$. § 110

$\therefore CF = FG$,

and $DC = BG$. § 128

$\therefore AG = AB - BG = AB - DC$.

But $EF = \frac{1}{2}AG$. § 189

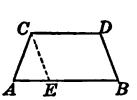
$\therefore EF = \frac{1}{2}(AB - DC)$. Q. E. D.

Ex. 84. *In an isosceles trapezoid each base makes equal angles with the legs.*

Let $ABDC$ be a trapezoid with leg AC equal to leg BD.

To prove that $\angle A = \angle B$, and $\angle C = \angle D$.

Proof. Draw $CE \parallel$ to DB.



$CE = DB$. § 180

$\therefore CE = AC$. Ax. 1

$\therefore \angle A = \angle CEA$. § 145

But $\angle CEA = \angle B$. § 112

$\therefore \angle A = \angle B$. Ax. 1

Now $\angle C$ is the supplement of $\angle A$,

and $\angle D$ is the supplement of $\angle B$. § 115

$\therefore \angle C = \angle D$. § 85

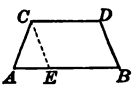
Q. E. D.

Ex. 85. *If the angles at the base of a trapezoid are equal, the other angles are equal, and the trapezoid is isosceles.*

Let $ABDC$ be a trapezoid having $AB \parallel$ to CD and $\angle A$ equal to $\angle B$.

To prove that $\angle D = \angle C$, and $AC = BD$.

Proof. The $\measuredangle D$ and C are equal, because they are the supplements of the equal $\measuredangle A$ and B, respectively. § 85

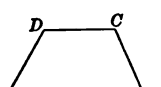


Draw	$CE \parallel$ to DB.	
Then	$\angle CEA = \angle B$.	§ 112
But	$\angle B = \angle A$.	Hyp.
	$\therefore \angle CEA = \angle A$.	Ax. 1
	$\therefore AC = CE$.	§ 147
But	$CE = BD$.	§ 180
	$\therefore AC = BD$.	Ax. 1

Q. E. D.

Ex. 86. *In an isosceles trapezoid the opposite angles are supplementary.*

Let $ABCD$ be an isosceles trapezoid, having $AB \parallel$ to DC, and AD equal to BC.



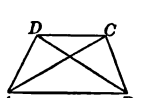
To prove that $\angle A + \angle C = 180° = \angle B + \angle D$.

Proof.	$\angle A + \angle D = 180°$,	§ 115
and	$\angle C = \angle D$.	Ex. 84
	$\therefore \angle A + \angle C = 180°$.	
Similarly,	$\angle B + \angle D = 180°$.	

Q. E. D.

Ex. 87. *The diagonals of an isosceles trapezoid are equal.*

Let $ABCD$ be an isosceles trapezoid, and AB, DC the $\parallel$ sides, AC, BD the diagonals.



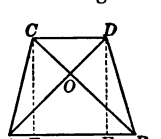
To prove that $AC = BD$.

Proof.	The $\triangle ADC$ and BCD are equal.	§ 143
For	DC is common,	
	$AD = BC$,	Hyp.
and	$\angle ADC = \angle BCD$.	Ex. 84
	$\therefore AC = BD$.	§ 128

Q. E. D.

Ex. 88. *If the diagonals of a trapezoid are equal, the trapezoid is isosceles.*

Let $ABCD$ be a trapezoid, AB, CD the $\parallel$ sides, and let the diagonals AD, BC be equal.



To prove that $AC = BD$.

Proof. From C, D drop the $\perp$s CE, DF to AB.

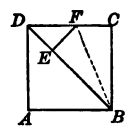
	CE is $\parallel$ to DF.	§ 104
	$\therefore CE = DF$.	§ 180
The rt. $\triangle ADF$ and BCE are equal.		§ 151

For $DF = CE$, and $AD = BC$. Hyp.
$\therefore \angle DAF = \angle CBA$. § 128
The $\triangle ABC$ and ABD are equal. § 143
For AB is common,
$BC = AD$, Hyp.
and $\angle CBA = \angle DAF$. § 128
$\therefore AC = BD$. Q. E. D.

Ex. 89. *If from the diagonal DB of a square ABCD, BE is cut off equal to BC, and EF is drawn perpendicular to BD meeting DC at F, then DE is equal to EF, and also to FC.*

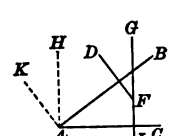
To prove that $DE = EF = FC$.

Proof. In the $\triangle BCD$, $\angle D = \angle B$. § 145
But $\angle D + \angle B = 90°$. $\therefore \angle D = 45°$.
In $\triangle DEF$, $\angle E + \angle D = 90° + 45° = 135°$.
$\therefore \angle F = 180° - 135° = 45°$.
$\therefore \triangle DEF$ is isosceles, and $DE = EF$. § 147
Draw BF. The rt. $\triangle BEF$ and BCF are equal. § 151
For BF is common,
and $BE = BC$. Hyp.
$\therefore EF = FC$. § 128
$\therefore DE = EF = FC$. Ax. 1
Q. E. D.



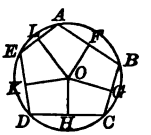
Ex. 90. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*

Let AB be perpendicular to FD, and AC to GI.
To prove that $\angle BAC$ is equal to $\angle DFG$, and supplementary to $\angle DFI$.
Proof. Draw $AK \perp$ to AB, and $AH \perp$ to AC.
Then AK is $\parallel$ to FD, and AH to IG. § 104
$\therefore \angle DFG = \angle KAH$. § 176
The $\angle BAK$ is a right angle by construction.
$\therefore \angle BAH$ is the complement of $\angle KAH$.
The $\angle CAH$ is a right angle by construction.
$\therefore \angle BAH$ is the complement of $\angle BAC$.
$\therefore \angle BAC = \angle KAH$. § 84
$\therefore \angle DFG = \angle BAC$. Ax. 1
$\therefore \angle DFI$, the supplement of $\angle DFG$, is also the supplement of $\angle BAC$.
Q. E. D.



Ex. 91. *The perpendicular bisectors of the sides of an inscribed polygon are concurrent (pass through the same point).*

Let $ABCDE$ be an inscribed polygon.



To prove that the ⊥ bisectors of AB, BC, CD, DE, EA are concurrent.

Proof. The ⊥ bisector of each side of the inscribed polygon passes through the centre of the circle. § 248

∴ the ⊥ bisectors of all the sides intersect at the centre of the circle.

Q. E. D.

Ex. 92. *Describe the relative position of two circles if the line of centres:*

(1) *is greater than the sum of the radii;*

(2) *is equal to the sum of the radii;*

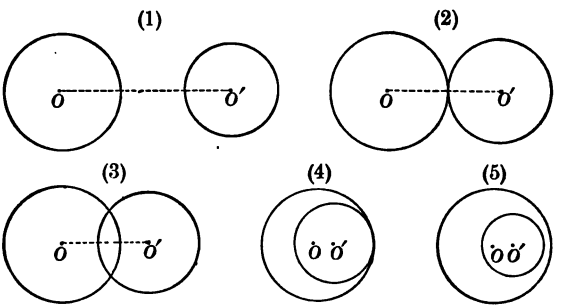
(3) *is less than the sum but greater than the difference of the radii;*

(4) *is equal to the difference of the radii;*

(5) *is less than the difference of the radii.*

Illustrate each case by a figure.

These five cases are illustrated by the following figures:



(1) The two Ⓢ are exterior to each other.

(2) The two Ⓢ are tangent to each other externally.

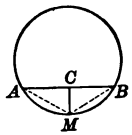
(3) The two Ⓢ intersect each other.

(4) The two Ⓢ are tangent to each other internally.

(5) The two Ⓢ lie one wholly within the other.

Ex. 93. *The straight line drawn from the middle point of a chord to the middle point of its subtended arc is perpendicular to the chord.*

Let C be the middle point of the chord AB, and let M be the middle point of the arc AB.



To prove that CM is $\perp$ to the chord AB.

Proof. Draw AM and BM.

The chord AM = the chord BM. § 241

$\therefore$ M is equidistant from A and B.

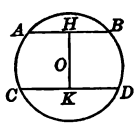
But C is equidistant from A and B. Hyp.

$\therefore$ CM is the $\perp$ bisector of AM. § 161

Q. E. D.

Ex. 94. *The line which passes through the middle points of two parallel chords passes through the centre of the circle.*

Let the chords AB and CD be parallel, let H be the middle point of AB, and K be the middle point of CD.



To prove that HK passes through the centre of the circle.

Proof. Draw OH and OK.

OH is $\perp$ to AB and OK is $\perp$ to CD. § 247

$\therefore$ OH and OK coincide and form one straight line.

$\therefore$ the line HK coincides with this line. § 47

$\therefore$ HK passes through the centre of the circle.

Q. E. D.

Ex. 95. *Describe the relative position of two circles if they may have:*

(1) *two common external and two common internal tangents;*

(2) *two common external tangents and one common internal tangent;*

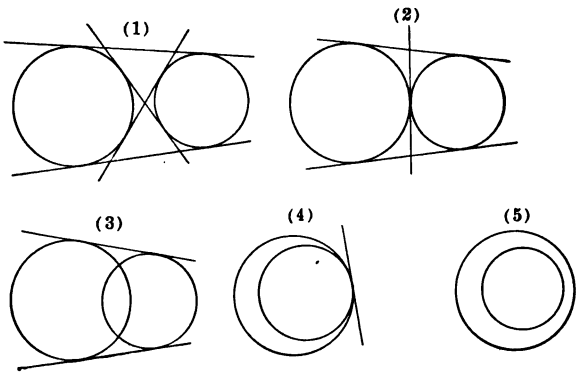
(3) *two common external tangents and no common internal tangent;*

(4) *one common external and no common internal tangent;*

(5) *no common tangent.*

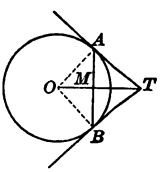
Illustrate each case by a figure.

These five cases are illustrated by the following figures:



(1) The two ⓢ are exterior to each other.
(2) The two ⓢ are tangent to each other externally.
(3) The two ⓢ intersect each other.
(4) The two ⓢ are tangent to each other internally.
(5) The two ⓢ lie one wholly within the other.

Ex. 96. *The line drawn from the centre of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.*



Let TA and TB be tangents to the circle whose centre is O.

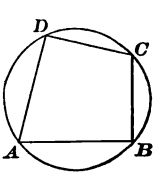
To prove that OT is the $\perp$ bisector of the chord AB.

Proof.

$TA = TB$.	§ 261
$OA = OB$.	§ 217
$\therefore$ OT is the $\perp$ bisector of AB.	§ 161

Q. E. D.

Ex. 97. *The opposite angles of an inscribed quadrilateral are supplementary.*



Let $ABCD$ be an inscribed quadrilateral.

To prove that $\angle A + \angle C = \angle B + \angle D = 180°$.

Proof. $\angle A$ is measured by $\frac{1}{2}$ arc DCB,

and $\angle C$ is measured by $\frac{1}{2}$ arc DAB.

$\therefore$ the sum, $\angle A + \angle C$, is measured by

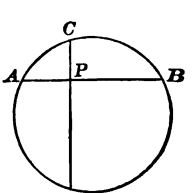
$\frac{1}{2}$ (arc DCB + arc DAB), or $\frac{1}{2}$ the circumference.

$\therefore \angle A + \angle C = \frac{1}{2}$ of $360° = 180°$.

Similarly, $\angle B + \angle D = 180°$.

Q. E. D.

Ex. 98. *If through a point within a circle two perpendicular chords are drawn, the sum of either pair of the opposite arcs which they intercept is equal to a semicircumference.*



Let AB and CD be two perpendicular chords intersecting at P.

To prove that

arc AC + arc BD = a semicircumference.

Proof. $\frac{1}{2}$ (arc AC + arc BD) measures $\angle APC$. § 294

But $\angle APC = 90°$. Hyp.

$\therefore \frac{1}{2}$ (arc AC + arc BD) $= 90°$.

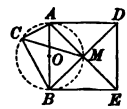
$\therefore$ arc AC + arc BD $= 180°$

= a semicircumference.

Q. E. D.

Ex. 99. *The line joining the centre of the square described upon the hypotenuse of a right triangle to the vertex of the right angle bisects the right angle.*

Let ABC be a rt. $\triangle$, and $\angle C$ a rt. $\angle$; also let M be the centre of the square $ABED$ constructed upon the hypotenuse AB; in other words, let M be the intersection of the two diagonals.



To prove that $\angle ACM = \angle BCM$.

Proof. Describe a $\odot$ upon AB as diameter.

$\angle ACB = 90°$, Hyp.

and $\angle AMB = 90°$. Ex. 72

$\therefore$ the $\odot$ will pass through C and M. § 290

Now $AM = BM$. Ex. 69, § 184

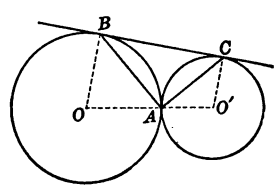
$\therefore$ arc AM = arc BM. § 243

$\therefore \angle ACM = \angle BCM$. § 289

$\therefore \angle ACM = \angle BCM = 45°$. Q. E. D.

Ex. 100. *Two circles are tangent externally at A, and a common external tangent touches them at B and C, respectively. Show that angle BAC is a right angle.*

Let BC be a common external tangent to the circles whose centres are O and O' and which are tangent externally at A.



To prove that $\angle BAC$ is a rt. $\angle$.

Proof. Draw OB, $O'C$, OO'.

Now OB and $O'C$ are both $\perp$ to BC, § 254

and are therefore parallel. § 104

$\therefore \angle O + \angle O' = 2$ rt. $\measuredangle$. § 115

$\angle O$ is measured by arc AB,

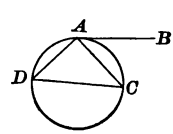
and $\angle O'$ is measured by arc AC. § 288

But $\angle BAC$ is measured by $\frac{1}{2}$ (arc AB + arc AC).

$\therefore \angle BAC$ is a rt. $\angle$. Q. E. D.

Ex. 101. *An angle formed by a tangent and a chord is equal to the angle inscribed in the opposite segment.*

Let the $\angle BAC$ be formed by the tangent AB and the chord AC; let ADC be any angle inscribed in the opposite segment.



To prove that $\angle BAC = \angle ADC$.

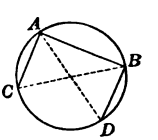
Proof. $\angle BAC$ is measured by $\frac{1}{2}$ arc AC. § 295

$\angle ADC$ is measured by $\frac{1}{2}$ arc AC. § 289

$\therefore \angle BAC = \angle ADC$. Ax. 1

Q. E. D.

Ex. 102. *Two chords drawn perpendicular to a third chord at its extremities are equal.*



Let AC and BD be chords $\perp$ to the chord AB at its extremities A and B.

To prove that $AC = BD$.

Proof. Draw AD and CB.

The rt. $\triangle CAB$ and DBA are equal. § 142

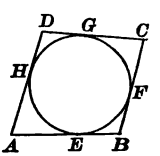
For $AB = AB$, Iden.

and $\angle C = \angle D$. § 293

$\therefore AC = BD$. § 128

Q. E. D.

Ex. 103. *The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.*



Let $ABCD$ be a circumscribed quadrilateral, and let E, F, G, H be the points of contact.

To prove that $AB + DC = AD + BC$.

Proof.

$AE = AH$, § 261

$BE = BF$,

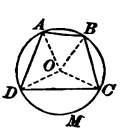
$CG = CF$,

$DG = DH$.

$\therefore AE + BE + CG + DG = AH + DH + BF + CF$. Ax. 2

That is, $AB + DC = AD + BC$. Q. E. D.

Ex. 104. *If the sum of two opposite angles of a quadrilateral is equal to two right angles, a circle may be circumscribed about the quadrilateral.*



In the quadrilateral $ABCD$, let $\angle DAB + \angle BCD$ be equal to 2 rt. $\triangle$. Let DAB be the circumference of a circle through D, A, and B.

To prove that the circumference DAB passes through C.

Proof. $\angle DAB$ is measured by $\frac{1}{2}$ arc DMB. § 289

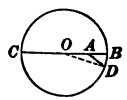
Since $\angle DAB + \angle BCD = 2$ rt. $\triangle$, Hyp.

$\angle BCD$ must be equal to an angle that is measured by $\frac{1}{2}$ the whole circumference $-$ arc DMB; that is, by $\frac{1}{2}$ arc DAB.

In order that $\angle BCD$ may be measured by $\frac{1}{2}$ arc DAB, C must lie on the circumference. Q. E. D.

Ex. 105. *The shortest line that can be drawn from a point within a circle to the circumference is the shorter segment of the diameter through that point.*

Let CB be the diameter of the circle CBD through the point A, and let AD be any other line from A to the circumference.

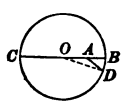


To prove that	$AB < AD$.	
Proof.	Draw OD.	
Now	$OD < OA + AD$.	§ 138
But	$OD = OB$.	§ 217
	$\therefore OB < OA + AD$,	
or	$OA + AB < OA + AD$. $\therefore AB < AD$.	Ax. 5

Q. E. D.

Ex. 106. *The longest line that can be drawn from a point within a circle to the circumference is the longer segment of the diameter through that point.*

Let CB be the diameter of the circle CBD through the point A, and let AD be any other line from A to the circumference.

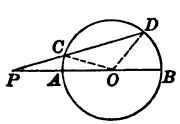


To prove that	$AC > AD$.	
Proof.	Draw OD.	
Now	$AO + OD > AD$.	§ 138
But	$OD = OC$.	§ 217
	$\therefore AO + OC > AD$.	
That is,	$AC > AD$.	

Q. E. D.

Ex. 107. *The shortest line that can be drawn from a point without a circle to the circumference will pass through the centre of the circle if produced.*

Let O be the centre of the given ⊙, P the given point, PAB a secant passing through O, PCD any other secant.

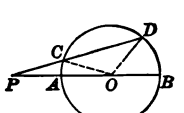


To prove that	$PA < PC$.	
Proof.	Draw OC, OD.	
Now	$PA + AO < PC + CO$,	§ 138
and	$AO = CO$.	§ 217
	$\therefore PA < PC$.	Ax. 5

Q. E. D.

Ex. 108. *The longest line that can be drawn from a point without a circle to the concave arc of the circumference passes through the centre of the circle.*

Let O be the centre of the given ⊙, P the given point, PAB a secant passing through O, PCD any other secant.

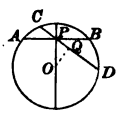


To prove that	$PB > PD$.	
Proof.	Draw OC, OD.	
Now	$PO + OD > PD$,	§ 138
and	$OD = OB$.	§ 217
	$\therefore PO + OB > PD$.	
That is,	$PB > PD$.	

Q. E. D.

Ex. 109. *The shortest chord that can be drawn through a point within a circle is perpendicular to the diameter at that point.*

Let P be the given point, O the centre, AB a chord passing through $P \perp$ to OP, CD any other chord through P.



To prove that $AB < CD$.

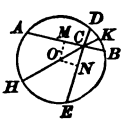
Proof. Draw $OQ \perp$ to CD, meeting CD at Q.

Now	$OP > OQ$.	§ 97
	$\therefore AB < CD$.	§ 251

Q. E. D.

Ex. 110. *If two intersecting chords make equal angles with the diameter drawn through the point of intersection, the two chords are equal.*

Let the two intersecting chords AB and DE make the equal angles ACH and ECH with the diameter HK drawn through their point of intersection C.



To prove that $AB = DE$.

Proof. Draw from O, the centre of the ⊙, $OM \perp$ to AB and $ON \perp$ to DE.

Rt. △ OMC and ONC are equal.		§ 141
For	$CO = CO$,	Iden.
and	$\angle MCO = \angle NCO$.	Hyp.
	$\therefore OM = ON$.	§ 128
	$\therefore AB = DE$.	§ 249

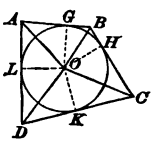
Q. E. D.

Ex. 111. *The angles subtended at the centre of a circle by any two opposite sides of a circumscribed quadrilateral are supplementary.*

Let $ABCD$ be a circumscribed quadrilateral, and from the centre of the circle let OA, OB, OC, OD be drawn.

To prove that $\angle AOD + \angle BOC = 2$ rt. $\angle$s.

Proof. Draw the radii OG, OH, OK, OL to the points of contact of the sides.



	The △ OGB and OHB are equal.	§ 150
For	$OB = OB$,	Iden.
	$OG = OH$,	§ 217
and	$BG = BH$.	§ 261
	$\therefore \angle GOB = \angle BOH$.	§ 128
Likewise	$\angle HOC = \angle COK$,	
	$\angle KOD = \angle DOL$,	
and	$\angle LOA = \angle AOG$.	

$\therefore \angle BOH + \angle HOC + \angle DOL + \angle LOA$

$= \angle GOB + \angle COK + \angle KOD + \angle AOG$. **Ax. 2**

$\therefore \angle BOC + \angle DOA = \angle AOB + \angle COD.$

$\therefore \angle AOD + \angle BOC = 2$ rt. $\measuredangle$,

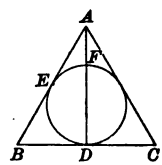
and $\angle AOB + \angle COD = 2$ rt. $\measuredangle$. § 73

Q. E. D.

Ex. 112. *The radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.*

Let the $\odot DEF$ be inscribed in the equilateral $\triangle ABC$.

To prove that the radius of the $\odot$ is equal to $\frac{1}{3}$ the altitude of the $\triangle$.



Proof. Since the $\odot$ is inscribed in the $\triangle$, the centre O is equidistant from the sides of the $\triangle$. § 217

Therefore, the centre is the point of intersection of the three bisectors of the $\measuredangle$ of the $\triangle$. § 162

Now the bisectors of the $\measuredangle$ of the $\triangle$ are $\perp$ to the opposite sides and bisect the opposite sides (Ex. 34), and are, therefore, the altitudes and the medians of the $\triangle$.

$\therefore OD = \frac{1}{3} AD.$ Ex. 27

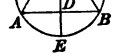
Q. E. D.

Ex. 113. *The radius of a circle circumscribed about an equilateral triangle is equal to two thirds the altitude of the triangle.*

Let the equilateral $\triangle ABC$ be inscribed in the $\odot ABC$.

To prove that the radius of the $\odot$ is equal to $\frac{2}{3}$ the altitude of the $\triangle$.

Proof. Since the $\odot$ is circumscribed about the $\triangle$, the centre O is equidistant from the three vertices. § 217



Therefore, the centre is the point of intersection of the $\perp$ bisectors of the sides of the $\triangle$. § 160

Now the $\perp$ bisectors of the sides of the $\triangle$ pass through the opposite vertices (Ex. 35), and are, therefore, the altitudes and the medians of the $\triangle$.

$\therefore OC = \frac{2}{3} CD.$ Ex. 27

Q. E. D.

Ex. 114. *A parallelogram inscribed in a circle is a rectangle.*

Let $ABCD$ be an inscribed parallelogram.

To prove that $ABCD$ is a rectangle.

Proof. $AB = DC$ and $BC = AD.$ § 178

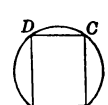
$\therefore$ arc $AB =$ arc DC and arc $BC =$ arc $AD.$ § 243

$\therefore$ arc $AB +$ arc $AD =$ arc $BC +$ arc $DC.$ Ax. 2

$\therefore$ arc $DAB =$ arc $BCD =$ arc $ABC =$ arc $ADC.$

$\therefore \angle A = \angle B = \angle C = \angle D = 90°.$

$\therefore ABCD$ is a rectangle. § 167

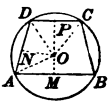


Q. E. D.

Ex. 115. *A trapezoid inscribed in a circle is an isosceles trapezoid.*

Let $ABCD$ be an inscribed trapezoid, having AB and DC ∥.

To prove that the trapezoid $ABCD$ is isosceles.



Proof. Now arc AD = arc BC. § 257

$\therefore AD = BC$. § 241

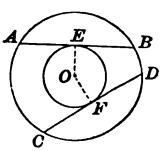
$\therefore$ trapezoid $ABCD$ is isosceles. § 173

Q. E. D.

Ex. 116. *All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.*

Let O be the common centre of the ⊙, AB, CD any two chords of the outer ⊙, touching the inner circle at the points E, F.

To prove that $AB = CD$, and $AE = BE$, $CF = DF$.



Proof. Draw OE, OF.

OE is ⊥ to AB, and OF is ⊥ to CD. § 254

$\therefore AB = CD$. § 249

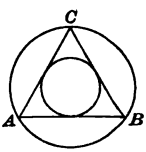
$\therefore AE = BE$, and $CF = DF$. § 245

Q. E. D.

Ex. 117. *If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.*

Let the inscribed and circumscribed circles of the $\triangle ABC$ be concentric.

To prove that the $\triangle ABC$ is equilateral.



Proof. $AB = AC = BC$. Ex. 116

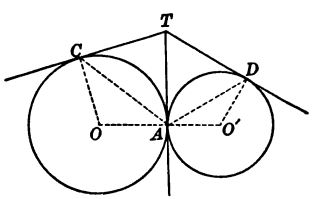
$\therefore \triangle ABC$ is equilateral. § 120

Q. E. D.

Ex. 118. *If two circles are tangent to each other, the tangents to them from any point of the common internal tangent are equal.*

Let the two circles be tangent to each other at A, and let AT be their common internal tangent.

From any point T of AT draw the tangents TC and TD.



To prove that $TC = TD$.

Proof. $TC = TA$, § 261

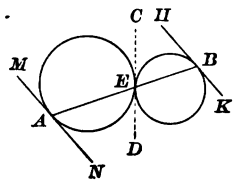
and $TD = TA$. § 261

$\therefore TC = TD$. Ax. 1

Q. E. D.

Ex. 119. *If two circles touch each other and a line is drawn through the point of contact terminated by the circumferences, the tangents at its ends are parallel.*

Let AB be a line drawn through the point of contact of two circles tangent to each other at the point E. Let MN and HK be tangents to the circles at the points A and B.



To prove that MN is $\parallel$ to HK.

Proof. Draw the common internal tangent CD.

$\angle HBE = \angle CEB$, § 295
(each being measured by $\frac{1}{2}$ arc EB).

$\angle NAE = \angle AED$, § 295
(each being measured by $\frac{1}{2}$ arc AE).

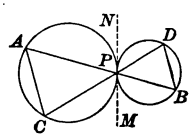
But $\angle CEB = \angle AED$. § 93

$\therefore \angle HBE = \angle NAE$. Ax. 1

$\therefore MN$ and HK are parallel. § 111

Q. E. D.

Ex. 120. *If two circles touch each other, and two lines are drawn through the point of contact terminated by the circumferences, the chords joining the ends of these lines are parallel.*



Let the ⓢ touch at P, and let AB, CD be lines through P terminated by the circumferences.

To prove that AC is $\parallel$ to DB.

Proof. Draw through P the common tangent MN.

Then $\angle A = \angle MPC$,

and $\angle B = \angle NPD$. Ex. 101

But $\angle MPC = \angle NPD$. § 93

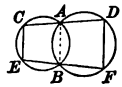
$\therefore \angle A = \angle B$. Ax. 1

$\therefore AC$ is $\parallel$ to DB. § 111

Q. E. D.

Ex. 121. *If two circles intersect and a line is drawn through each point of intersection terminated by the circumferences, the chords joining the ends of these lines are parallel.*

Let the ⓢ intersect in the points A, B, and let CAD, EBF be any two lines through A, B terminated by the circumferences.



To prove that CE is $\parallel$ to DF.

Proof. Draw AB.

$ACEB$, $ADFB$ are inscribed quadrilaterals.

$\therefore \angle E = 180° - \angle BAC$, BAC, Ex. 97

and $\angle F = 180° - \angle BAD$.

But $\angle BAC = 180° - \angle BAD$. § 86

$\therefore \angle BAC = \angle F$, and $\angle E = 180° - \angle F$.

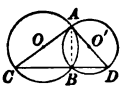
$\therefore \angle E + \angle F = 180°$, and CE is $\parallel$ to DF. § 116

Q. E. D.

Ex. 122. *Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.*

Let O, O' be the centres of the ⊙, A and B the points of intersection, AC, AD the two diameters.

To prove that the straight line CD passes through B.



Proof. Draw AB, BC, BD.

The $\measuredangle ABC$, ABD are rt. $\measuredangle$. § 290

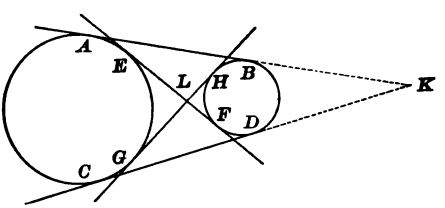
$\therefore CBD$ is a straight line. § 90

$\therefore CD$ passes through B. § 47

Q. E. D.

Ex. 123. *If two common external tangents or two common internal tangents are drawn to two circles, the segments intercepted between the points of contact are equal.*

Let AB and CD be the two common external tangents, and EF and GH the two common internal tangents to the two circles.



To prove that $AB = CD$, and $EF = GH$.

Proof. Produce AB and CD to meet at K.

Then $KA = KC$, and $KB = KD$. § 261

$\therefore KA - KB = KC - KD$. Ax. 3

That is, $AB = CD$.

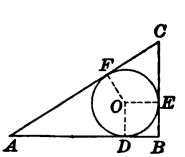
Again, $LH = LF$, and $LG = LE$. § 261

$\therefore LH + LG = LF + LE$. Ax. 2

That is, $HG = EF$. Q. E. D.

Ex. 124. *The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.*

Let ABC be a rt. $\triangle$, $\angle B = 90°$, O the centre of the inscribed ⊙, D, E, F the points of contact.



To prove that diameter of ⊙ $= AB + BC - AC$.

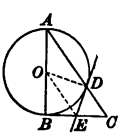
Proof. Draw the radii OD, OE.

OD is $\perp$ to AB, and OE is $\perp$ to BC. § 254

$\therefore OD$ is $\parallel$ to CB, and OE is $\parallel$ to AB. § 104
$\therefore OD = BE$, and $OE = BD$. § 180
$\therefore OD + OE = BE + BD$ Ax. 2
$= (BC - EC) + (AB - AD)$
$= BC - FC + AB - AF$ § 261
$= AB + BC - (AF + FC)$.
$\therefore$ diameter of $\odot = AB + BC - AC$. Q. E. D.

Ex. 125. *If one leg of a right triangle is the diameter of a circle, the tangent at the point where the circumference cuts the hypotenuse bisects the other leg.*

Let the diameter AB of the circle ADB be the leg AB of the rt. $\triangle ABC$, and let the hypotenuse AC cut the circumference at D. Let DE be the tangent at D, cutting the leg BC at E.

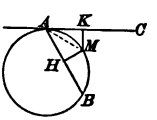


To prove that $EB = EC$.

Proof. Draw OE and OD.
$\angle BOE = \angle DOE$. § 261
$\angle BOD = \angle OAD + \angle ODA$. § 137
$\angle OAD = \angle ODA$. § 145
$\therefore \angle BOE = \angle OAD$. Ax. 7
$\therefore OE$ and AC are $\parallel$. § 114
Since $OA = OB$, $BE = EC$. § 188
Q. E. D.

Ex. 126. *If, from any point in the circumference of a circle, a chord and a tangent are drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal.*

Let the tangent AC and the chord AB be drawn from the same point A, and let M be the middle point of the arc AB.

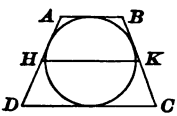


To prove that the $\perp MK = \perp MH$.

Proof. Draw AM.
$\angle BAM$ is measured by $\frac{1}{2}$ arc BM. § 289
$\angle CAM$ is measured by $\frac{1}{2}$ arc AM. § 295
But arc BM = arc AM. Hyp.
$\therefore \angle BAM = \angle CAM$. Ax. 7
Rt. $\triangle AHM$ and AKM are equal. § 141
For $\angle MAH = \angle KAM$, and AM is common.
$\therefore MH = MK$. § 128
Q. E. D.

Ex. 127. *The median of a trapezoid circumscribed about a circle is equal to one fourth the perimeter of the trapezoid.*

Let HK be the median of the circumscribed trapezoid $ABCD$.



To prove that $HK = \frac{1}{2}(AB + BC + CD + DA)$.

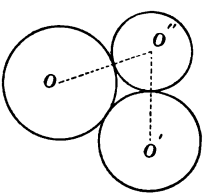
Proof. $AB + CD = BC + DA$. Ex. 103

But $HK = \frac{1}{2}(AB + CD)$. § 190

$\therefore HK = \frac{1}{2}(AB + BC + CD + DA)$.

Q. E. D.

Ex. 128. *Two fixed circles touch each other externally and a circle of variable radius touches both externally. Show that the difference of the distances from the centre of the variable circle to the centres of the fixed circles is constant.*



Let r and r' be the radii of two fixed circles that are tangent to each other externally. Let r'' be the variable radius of a circle that touches both externally. Let r be greater than r'.

To prove that the difference of the distances from the centre of the variable circle to the centres of the fixed circles is constant.

Proof. Draw the two lines of centres.

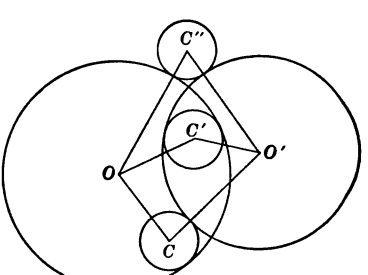
These lines of centres pass through the points of contact. § 265

$\therefore r + r'' - (r' + r'')$ is the required difference.

Now $r + r'' - (r' + r'') = r + r'' - r' - r'' = r - r'$, a constant.

Q. E. D.

Ex. 129. *If two fixed circles intersect, and circles are drawn to touch both, show that either the sum or the difference of the distances of their centres from the centres of the fixed circles is constant, according as they touch (i) one internally and one externally, (ii) both internally or both externally.*



Let O and O' be the centres and r and r' the radii of the two fixed ⑤. Let C be the centre and s the radius of the variable ⊙ tangent to one fixed ⊙ internally and to the other externally. Let C' be the centre and s' the radius of the variable ⊙ tangent to both the fixed ⑤ internally. Let C'' be the centre and s'' the radius of the variable ⊙ tangent to both the fixed ⑤ externally.

Draw the lines of centres OC, OC', OC'', $O'C$, $O'C'$, $O'C''$.

Let r be greater than r'.

To prove that $OC + O'C$, $OC' - O'C'$, $OC'' - O'C''$ are constant.

Proof. The lines of centres pass through the points of contact. § 265

$\therefore OC + O'C = (r - s) + (r' + s) = r - s + r' + s = r + r'$, a constant;

$O'C - O'C' = (r - s') - (r' - s') = r - s' - r' + s' = r - r'$, a constant;

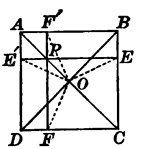
and $OC'' - O'C'' = (r + s'') - (r' + s'') = r + s'' - r - s'' = r - r'$, a constant.

Q. E. D.

Ex. 130. *If two straight lines are drawn through any point in a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides lie on the circumference of a circle whose centre is the point of intersection of the diagonals.*

Let P be any point in the diagonal AC of the square $ABCD$, and let EE' and FF' be drawn through $P \parallel$ to BA and DA, respectively.

To prove that $OE = OE' = OF = OF'$.



Proof. PE is $\perp$ to BC and PF is $\perp$ to DC.

Since P is in the bisector of $\angle DCB$, Ex. 72

$PE = PF$. § 162

Now $\angle CPE = \angle CAB$,

and $\angle CPF = \angle CAD$. § 177

$\therefore \angle CPE = \angle CPF$. Ax. 1

$PE = PF$, $PO = PO$. Iden.

$\therefore \triangle OPE = \triangle OPF$. § 143

$\therefore OE = OF$. § 128

In like manner, $OE' = OF'$.

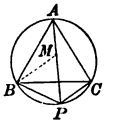
Since O is in the $\perp$ bisector of AB and EE', § 160

$OE = OE'$. § 95

$\therefore OE = OE' = OF = OF'$. Q. E. D.

Ex. 131. *If ABC is an inscribed equilateral triangle and P is any point in the arc BC, then $PA = PB + PC$.*

Proof. Upon PA take PM equal to PB, and draw BM.



$\angle BPM = 60°$. § 289

$\therefore \triangle BPM$ is equilateral. Ex. 77

In the $\triangle ABM$, BCP,

$AB = BC$, and $BM = BP$.

Also $\angle ABM = \angle ABP - \angle MBP = \angle ABP - 60°$.

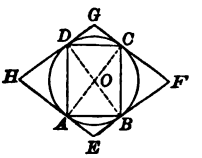
$\angle PBC = \angle ABP - \angle ABC = \angle ABP - 60°$.

$\therefore \angle ABM = \angle PBC$.

$\therefore \triangle ABM = \triangle BCP$ (§ 150), and $AM = PC$.

$\therefore PA = PM + MA = PB + PC$. Q. E. D.

Ex. 132. *The tangents drawn through the vertices of an inscribed rectangle, which is not a square, enclose a rhombus.*



Let O be the centre, $ABCD$ the inscribed rectangle, and let the tangents drawn through A, B, C, and D meet in the points E, F, G, and H.

To prove that the figure $EFGH$ is a rhombus.

Proof. The four $\triangle$ in the figure are isosceles. § 261

Since $AD = BC$, § 178

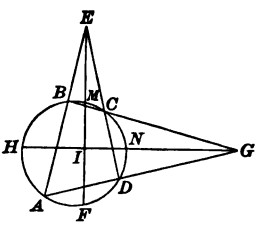
arc AD = arc BC. § 241

And $\angle A = \angle D = \angle B = \angle C$. § 295

$\therefore \triangle AHD = \triangle BFC$. § 139

Q. E. D.

Ex. 133. *The bisectors of the angles included by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.*



Let $ABCD$ be the inscribed quadrilateral, and let AB and DC meet at E, and AD and BC meet at G. Also let the bisectors EF, GH of the $\triangle$ E and G meet at I.

To prove that EF is $\perp$ to GH.

Proof. Let the bisector of $\angle E$ cut the $\odot$ at M and F, and the bisector of the $\angle G$ cut the $\odot$ at N and H.

Then, since $\angle AEF = \angle DEF$, Hyp.

$\frac{1}{2}$(arc AF − arc BM) = $\frac{1}{2}$(arc DF − arc CM). § 296

$\therefore$ arc AF − arc BM = arc DF − arc CM. Ax. 6

Similarly, arc AH − arc DN = arc BH − arc CN.

$\therefore$ arc FH − arc BM − arc DN = arc DF + arc BH − arc MN. Ax. 2

That is, arc FH + arc MN = arc HM + arc FN.

Hence, the measure of the $\triangle$ FIH, HIM is the same. § 294

$\therefore \angle FIH = \angle HIM = 90°$. § 63

$\therefore EF$ is $\perp$ to GH. Q. E. D.

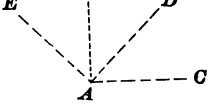
Discussion. This problem is impossible, if any two sides of the quadrilateral are parallel.

Ex. 134. *To construct an angle of* 45°; *of* 135°.

(1) Construct a rt. $\angle BAC$. § 301

Draw AD bisecting the $\angle BAC$. § 304

Then $\angle DAC$ is the $\angle$ required.



Proof. $\angle BAD = \angle DAC = 45°$.

(2) Draw $AE \perp$ to AD. § 301

Then $\angle CAE$ is the $\angle$ required.

Proof. $\angle CAE = 90° + 45° = 135°$. Q. E. F.

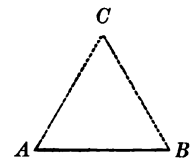
Ex. 135. *To construct an equilateral triangle, having given one side.*

Let AB be the given side.

With centres A and B and radius AB, describe arcs intersecting in C.

Draw AC, BC.

Then $\triangle ABC$ is the triangle required, as is obvious from the construction. Q. E. F.

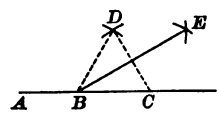


Ex. 136. *To construct an angle of* 60°; *of* 150°.

Draw a straight line ABC, and with two points in it, B, C, as centres, and BC for radius, describe arcs intersecting at D.

With centres D and C, and the same radius, describe arcs intersecting at E. Draw BE.

$\angle DBC = 60°$, and $\angle ABE = 150°$.



Proof. Draw BD, CD.

$\triangle BCD$ is equilateral. Const.

$\therefore \angle DBC = 60°$. § 146

$\therefore \angle ABD = 180° - 60° = 120°$. § 86

$\angle DBE = \frac{1}{2} \angle DBC = 30°$. § 304

$\therefore \angle ABE = 120° + 30° = 150°$. Q. E. F.

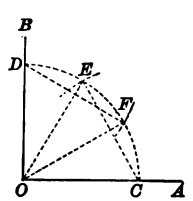
Ex. 137. *To trisect a right angle.*

Let $\angle AOB$ be a right angle.

With centre O and any radius, describe an arc cutting OA at C, and OB at D.

With centres C and D, and the same radius as before, cut arc DC at E and F, respectively.

Then $\angle COF = \angle FOE = \angle EOD = 30°$.



Proof. $\triangle OCE$, OFD are equilateral by construction.

$\therefore \angle COE = 60°$, § 146

Similarly, $\angle COF = 30°$.

$\therefore \angle EOD = 90° - \angle COE = 30°$.

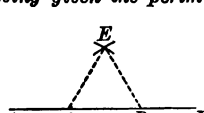
$\therefore \angle FOE = 90° - 60° = 30°$. Q. E. F.

Ex. 138. *To construct an equilateral triangle, having given the perimeter.*

Let AB be the given perimeter.

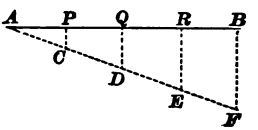
Trisect AB at the points C, D. § 307

With centres C and D, and radius AC, describe arcs intersecting at E.



Proof. Draw CE, DE. $\triangle CDE$ is the equilateral $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 139. *To divide a line into four equal parts by two different methods.*

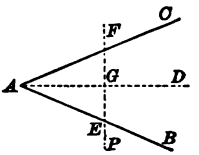


Let AB be the given line.

First Method. Draw AF and apply § 307.

Second Method. Bisect the line at Q, § 302, and then bisect AQ at P, BQ at R. Q. E. F.

Ex. 140. *Through a given point to draw a line which shall make equal angles with the two sides of a given angle.*



Let P be the point, BAC the angle.

Draw AD bisecting the $\angle BAC$. § 304

Through P draw $EF \perp$ to AD, meeting AB at E, AC at F, AD at G. § 300

EF is the line required.

Proof. $\triangle AEG = \triangle AFG$. § 142

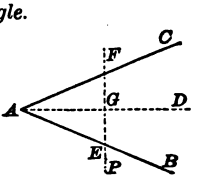
For AG is common,

and $\angle EAG = \angle FAG$. Const.

$\therefore \angle AEG = AFG$. § 128

Q. E. F.

Ex. 141. *To draw a line through a given point, so that it shall form with the sides of a given angle an isosceles triangle.*



Let P be the point, BAC the angle.

Draw AD bisecting the $\angle BAC$.

Through P draw $EF \perp$ to AD, meeting AB at E, AC at F.

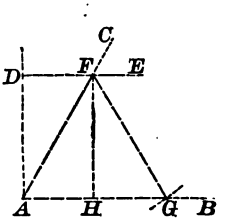
EF is the line required.

Proof. $\angle AEF = \angle AFE$. Ex. 140

$\therefore AF = AE$. § 147

Q. E. F.

Ex. 142. *To construct an equilateral triangle, having given the altitude.*



Construct an $\angle BAC$ equal to 60°. Ex. 136

Erect at A a $\perp AD$ equal to the given altitude.

Draw $DE \parallel$ to AB, meeting AC at F.

With centre F and radius FA cut AB at G.

Draw FG. $\triangle AFG$ is the $\triangle$ required.

Proof. The altitude is equal to the given altitude. Const.

The $\triangle AFG$ is isosceles, Const.

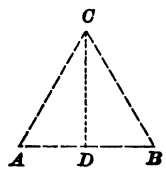
and has one $\angle A$ equal to 60°; Const.

$\therefore \triangle AFG$ is equiangular. Ex. 9

$\therefore \triangle AFG$ is equilateral. § 148

Q. E. F.

Ex. 143. *To construct an isosceles triangle, having given the base and the altitude.*



Take AB equal to the given base.

At D, the middle point of the base, erect a perpendicular DC equal to the given altitude.

Draw CA and CB. $\triangle ABC$ is the $\triangle$ required.

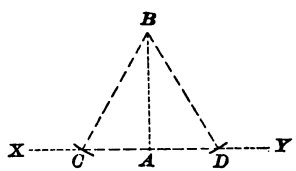
Proof. AB is equal to the given base, Const.

CD is equal to the given altitude, Const.

and $CA = CB$. § 160

Q. E. F.

Ex. 144. *To construct an isosceles triangle, having given the altitude and one of the legs.*



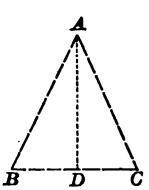
At any point A in the line XY erect a perpendicular AB equal to the given altitude.

With B as a centre and a radius equal to the given leg, describe arcs intersecting XY at C and D.

Draw BC and BD.

$\triangle BCD$ is the $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 145. *To construct an isosceles triangle, having given the angle at the vertex and the altitude.*



Construct $\angle BAC$ equal to the given $\angle$.

Draw AD bisecting $\angle BAC$. § 304

Take AD equal to the given altitude, and erect at D a $\perp$ meeting the sides of the $\angle$ at B and C. § 301

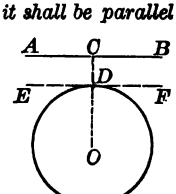
$\triangle ABC$ is the $\triangle$ required.

Proof. The $\triangle ABC$ has the given parts by the construction; also, $\triangle ACD = \triangle BCD$. § 142

$\therefore AC = BC$. § 128

Q. E. F.

Ex. 146. *To draw a tangent to a given circle, so that it shall be parallel to a given straight line.*



Let O be the centre of the given $\odot$, AB the given line.

Draw $OC \perp AB$, meeting the $\odot$ at D.

Through D draw $EF \perp$ to OC.

Then EF is the tangent required.

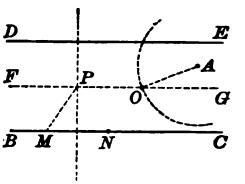
Proof. EF is a tangent to the $\odot$. § 253

EF is $\parallel$ to AB § 104

Q. E. F.

Ex. 147. *To construct a circle that shall pass through a given point and cut chords of a given length from two parallels.*

Analysis. Suppose the problem solved. Let A be the given point, BC and DE the given parallels, MN the given length, and O the centre of the required circle.



Since the circle cuts equal chords from two parallels, its centre must be equidistant from them. Therefore, one locus for O is $FG \parallel$ to BC and equidistant from BC and DE.

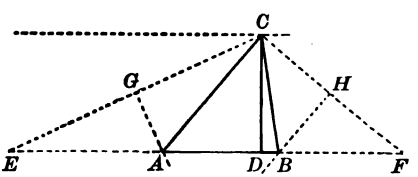
Draw the $\perp$ bisector of MN, cutting FG in P. PM is the radius of the circle required. With A as centre and radius PM describe an arc cutting FG at O.

Then O is the centre of the required circle. Q. E. F.

Discussion. The problem is impossible if the distance from A to FG is greater than PM.

Ex. 148. *To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle.*

Analysis. Suppose the problem solved, and let ABC be the $\triangle$ required, ACB the given $\angle$, and CD the given altitude.



Produce AB both ways, and take AE equal to AC, and BF equal to BC.

Then EF = the given perimeter.

Draw CE and CF, forming the isosceles $\triangle$ CAE and CBF.

In the $\triangle ECF$, $\quad \angle E + \angle F + \angle ECF = 180°$. § 129

But $\quad \angle ECF = \angle ECA + \angle FCB + \angle ACB$. Ax. 9

Since $\quad \angle E = \angle ECA$ and $\angle F = \angle FCB$, § 145

we have $\quad \angle ECF = \angle E + \angle F + \angle ACB$.

$\therefore 2\angle E + 2\angle F + \angle ACB = 180°$.

$\therefore \angle E + \angle F + \frac{1}{2}\angle ACB = 90°$,

and $\quad \angle E + \angle F = 90° - \frac{1}{2}\angle ACB$.

By substitution, $\angle ECF = 90° + \frac{1}{2}\angle ACB$.

$\therefore \angle ECF$ is known.

Construction. To find the point C, construct on EF a segment that will contain the $\angle ECF$. § 318

Draw a parallel to EF at the distance CD, the given altitude.

To find the points A and B, draw the $\perp$ bisectors of the lines CE and CF, and the points A and B will be vertices of the required $\triangle$. Q. E. F.

Ex. 149. *Find the locus of a point at a given distance from a given circumference.*

Let r denote the radius of the given circumference, O its centre, d the given distance. Then the locus consists of the two circumferences, having for centre the point O, and for radii $r + d$ and $r - d$, respectively. Q. E. F.

Discussion. If $d = r$, the second of these circumferences becomes the point O; if $d > r$, the second circumference does not exist.

Ex. 150. *Find the locus of the centre of a circle which has a given radius r and passes through a given point P.*

The required locus is the circumference of a $\odot$ having P for centre and r for radius. Q. E. F.

Ex. 151. *Find the locus of the centre of a circle which has a given radius r and touches a given line AB.*

The required locus consists of the two straight lines drawn $\parallel$ to AB, one on one side of AB, the other on the other side, at the distance r from AB. Q. E. F.

Ex. 152. *Find the locus of the centre of a circle which passes through two given points P and Q.*

The locus is the $\perp$ bisector of the line joining the given points. § 248 Q. E. F.

Ex. 153. *Find the locus of the centre of a circle which touches a given straight line AB at a given point P.*

The required locus is the $\perp$ to AB erected at P. § 255 Q. E. F.

Ex. 154. *Find the locus of the centre of a circle which touches each of two given parallels.*

This locus is the straight line parallel to the given parallels and midway between them. Q. E. F.

Ex. 155. *Find the locus of the centre of a circle which touches each of two given intersecting lines.*

The locus consists of the two bisectors of the $\measuredangle$ formed by the given lines. § 162 Q. E. F.

Ex. 156. *To find in a given line a point X which is equidistant from two given points.*

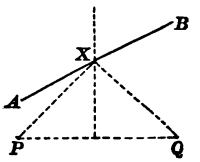
Let AB be the given line, P, Q the given points.

Draw PQ, and erect its perpendicular bisector, cutting AB at X. § 302

Then X is the point required.

Proof. $PX = QX$. § 160

Q. E. F.



Ex. 157. *To find a point X equidistant from three given points.*

Let A, B, C be the given points.

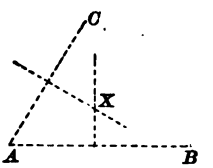
Draw AB, AC, and erect upon each line its perpendicular bisector.

Let these perpendicular bisectors meet at X.

Then X is the point required.

Proof. X is equidistant from the three given points A, B, C. § 160

Q. E. F.



Discussion. The problem is impossible if the points A, B, C are in a straight line.

Ex. 158. *To find a point X equidistant from two given points and at a given distance from a third given point.*

Let P, Q be the given points, d the given distance, R the third point.

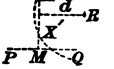
Draw PQ, and erect its perpendicular bisector MN.

With centre R and radius d, describe an arc cutting MN at X.

Then X is the point required.

Proof. X is equidistant from P and Q, § 160

and at the distance d from R. Const. Q. E. F.



Discussion. Let d' denote the distance from R to MN. Then the number of solutions is 2, 1, or 0, according as $d >$, $=$, or $< d'$.

In the figure there are two solutions, X and X'.

Ex. 159. *To construct a circle which has a given radius and passes through two given points.*

Let P, Q be the given points, r the given radius.

Draw PQ, and erect its perpendicular bisector MN.

With centre P and radius r, cut MN at O.

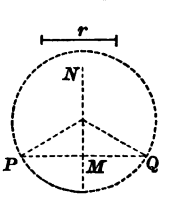
With centre O and radius r, describe a $\odot$.

This $\odot$ is the $\odot$ required.

Proof. The $\odot$ has r for radius, Const.

and passes through P and Q. § 160

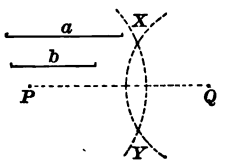
Q. E. F.



Discussion. The arc described with centre P and radius r will also cut MN in a point below PQ; hence two Ⓢ answering the conditions may be described. These Ⓢ, however, are equal, and differ merely in position. This problem is impossible if $r < \frac{1}{2} PQ$.

Ex. 160. *To find a point X at given distances from two given points.*

Let P, Q be the given points, and let a, b be the given distances.



With centre P and radius a, describe an arc.

With centre Q and radius b, describe another arc cutting the first arc at X.

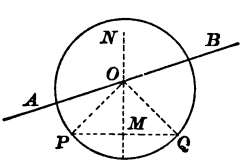
Then X is the point required.

Proof. X is, by construction, at the distance a from P, and at the distance b from Q. Q. E. F.

Discussion. If $PQ = a + b$, there is one solution; namely, a point in the line PQ where the two arcs touch each other. If $PQ > a + b$, or $< a - b$, the problem is impossible; for the arcs cannot cut each other and cannot touch each other. (See Ex. 92.) If PQ has any value between $a + b$ and $a - b$, there are two solutions (as in the figure).

Ex. 161. *To construct a circle which has its centre in a given line and passes through two given points.*

Let AB be the given line, P and Q the given points.



Draw PQ, and erect its perpendicular bisector MN. Let MN meet AB at O.

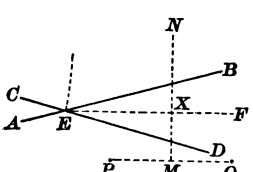
With centre O and radius OP, describe a ⊙. This ⊙ is the ⊙ required.

Proof. The ⊙ passes through P and Q. § 160

Q. E. F.

Ex. 162. *To find a point X equidistant from two given points and also equidistant from two given intersecting lines.*

Let P, Q be the given points, and let AB, CD be the given lines intersecting at E.



Draw PQ, and erect its perpendicular bisector MN.

Draw EF bisecting $\angle DEB$, meeting MN at X.

Then X is the point required.

Proof. X is equidistant from P and Q, § 160

and also equidistant from AB and CD. § 162

Q. E. F.

Discussion. By bisecting the $\angle BEC$, another solution Y may be obtained. Therefore, in general there are two solutions. If, however, EF coincides with PQ, or is $\parallel$ to PQ, the other bisector EY is $\parallel$ to MN (§§ 104, 107), and there is only one solution; namely, the point M. If EF coincides with MN, every point in MN is a solution.

Ex. 163. *To find a point X equidistant from two given points and also equidistant from two given parallel lines.*

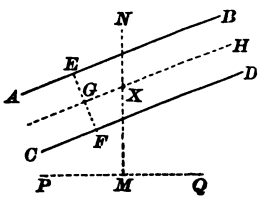
Let P, Q be the points, AB and CD the parallels.

Draw PQ, and erect its perpendicular bisector MN.

Draw $EF \perp$ to AB.

Bisect EF at G.

Draw $GH \parallel$ to AB, meeting MN at X.



Then X is the point required.

Proof. Every point in MN is equidistant from P and Q (§ 160), and every point in GH is equidistant from AB and CD. § 181

$\therefore$ X, the point common to MN and GH, is equidistant from P and Q, and also equidistant from AB and CD. Q. E. F.

Discussion. In general, there is one solution. If AB is $\perp$ to PQ, there is no solution, unless GH coincides with MN; and then every point in MN is a solution.

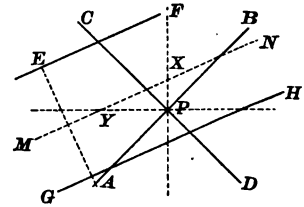
Ex. 164. *To find a point X equidistant from two given intersecting lines, and also equidistant from two given parallels.*

Let AB, CD be the intersecting lines, P the point of intersection, EF and GH the two parallel lines.

Draw $MN \parallel$ to EF and GH, and midway between them.

Also draw the bisectors of the angles formed at P.

Let these bisectors meet MN at X and Y.



Then X and Y are the points required.

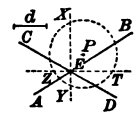
Proof. X and Y are equidistant from EF and GH because they are in MN, and equidistant from AB and CD because they are in the bisectors of the angles at P. § 162

Q. E. F.

Discussion. If MN passes through P, there is only one solution; namely, the point P. If MN is $\parallel$ to one of the bisectors, there is but one solution. If MN coincides with one of the bisectors, every point in MN is a solution.

Ex. 165. *To find a point X equidistant from two given intersecting lines and at a given distance from a given point.*

Let AB, CD be the given lines, intersecting at E, d the given distance, P the given point.



Draw the bisectors XY, TZ of the angles formed by the two lines AB, CD.

With centre P and radius d, describe an arc cutting the bisectors in X, Y, Z, T.

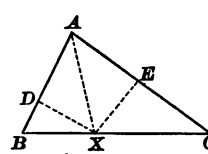
Then each of the points X, Y, Z, T satisfies the conditions.

Proof. Each of the points is equidistant from the given lines, and at the distance d from P by construction. § 162 Q. E. F.

Discussion. If (as in the figure) d is greater than the distance from P to each bisector, there are four solutions. If d is less than the distance from P to each bisector, there is no solution. Between these extremes there may be one, two, or three solutions.

Ex. 166. *To find a point X which lies in one side of a given triangle and is equidistant from the other two sides.*

Let ABC be the given triangle.



Bisect the $\angle A$, and let the bisector meet BC at X.

Then X is the point required.

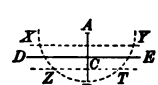
Proof. Draw $XD \perp$ to AB and $XE \perp$ to AC.

$XD = XE$. § 162

Q. E. F.

Ex. 167. *A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction how many places answer this description.*

Let A represent the town. With centre A and any convenient radius describe a circle.



Draw a radius AB; and take this length to represent 4 miles.

Construct the perpendicular bisector DE to the line AB, bisecting it at C.

Then DE represents the railroad.

Draw parallels to DE on either side, at a distance from DE equal to $\frac{1}{4}AC$ (1 mile).

Let these parallels cut the circle in the points X, Y, Z, T.

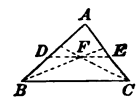
Then any one of these four points satisfies the conditions.

Therefore, there are four solutions. Q. E. F.

Ex. 168. *In a triangle ABC, to draw DE parallel to the base BC, cutting the sides of the triangle in D and E, so that DE shall equal $DB + EC$.*

Bisect $\measuredangle B$ and C, and let the bisectors meet at F.

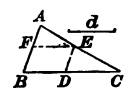
Draw through F a line $\parallel$ to BC, meeting AB at D, and AC at E.



Then	$DE = BD + EC$.	
Proof.	$\angle DFB = \angle FBC$.	§ 110
Also	$\angle DBF = \angle FBC$.	Const.
	$\therefore \angle DBF = \angle DFB$.	Ax. 1
	$\therefore DF = DB$.	§ 147
Similarly,	$FE = EC$.	
	$\therefore DF + FE = DB + EC$.	Ax. 2
That is,	$DE = BD + EC$.	Q. E. F.

Ex. 169. *To draw through two sides of a triangle a line parallel to the third side so that the part intercepted between the sides shall have a given length.*

Let ABC be the given $\triangle$, d the given length.



Upon BC take BD equal to d.

Draw $DE \parallel$ to BA, meeting AC at E.

Draw $EF \parallel$ to CB, meeting AB at F.

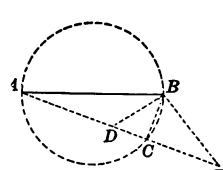
Then EF is the line required.

Proof.	EF is $\parallel$ to CB.	Const.
	The figure $BDEF$ is a $\square$.	§ 166
	$\therefore EF = BD$.	§ 178
But	$BD = d$.	Const.
	$\therefore EF = d$.	Ax. 1
		Q. E. F.

Ex. 170. *Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter.*

Let AB be the given hypotenuse.

With AB as diameter, describe a $\odot$.



Proof. 1. Join any point C of the circumference to A and B.

Then $\angle ACB = 90°$. § 290

2. Join any point D in AC to B.

Then $\angle ADB > \angle ACB$, or $90°$. § 137

Produce AC to any point E, and join EB.

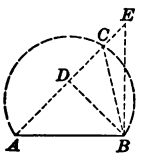
Then $\angle AEB < \angle ACB$, or $90°$. § 137

Therefore, the circumference with AB for diameter is the locus of the vertex of a rt. $\triangle$ having AB for hypotenuse. Q. E. F.

Ex. 171. *Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing the given angle.*

Let AB be the given base. Upon AB construct a segment capable of containing the given angle. § 318

Proof. 1. Join any point C in the arc of this segment to A and B.



Then $\angle ACB =$ the given angle.

2. Join any point D in AC to B.

Then $\angle ADB > \angle ACB$. § 137

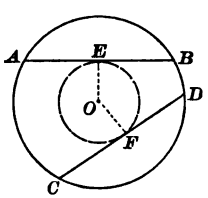
Produce AC to any point E, and join EB.

Then $\angle AEB < \angle ACB$. § 137

Therefore, the arc of a segment constructed upon AB as base and capable of containing the given angle is the locus of a vertex of a $\triangle$ having AB for base, and the given angle for the $\angle$ at the vertex. Q. E. F.

Ex. 172. *Find the locus of the middle point of a chord of a given length that can be drawn in a given circle.*

Let O be the centre of the given $\odot$, AB, CD two chords of the given length.



Proof. Draw $OE \perp$ to AB, $OF \perp$ to CD.

Then $AE = BE$, $CF = DF$. § 245

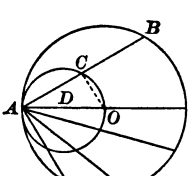
Also $OE = OF$. § 249

Similarly, the middle points of all chords of the given length are equidistant from O.

Therefore, the required locus is the circumference of a $\odot$ concentric with the given $\odot$, having for radius the distance from the centre of the given $\odot$ to any one of the chords. Q. E. F.

Ex. 173. *Find the locus of the middle point of a chord drawn from a given point in a given circumference.*

Let O be the centre of the given $\odot$, A any point in its circumference, AB any chord drawn through A, C its middle point.



Proof. Draw OC.

Then $\angle ACO = 90°$. § 247

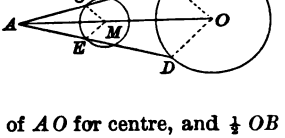
$\therefore$ C is in the circumference described upon AO as diameter. Ex. 170

Similarly, the middle point of every chord drawn from A lies in this circumference.

Therefore, this circumference is the required locus. Q. E. F.

Ex. 174. *Find the locus of the middle point of a straight line drawn from a given exterior point to a given circumference.*

Let O be the centre of the given ⊙, AB any one of the straight lines, C its middle point.



Proof. Draw OA, OB, bisect AO at M, and draw CM.

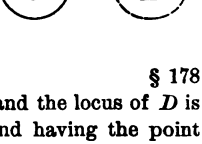
MC is ∥ to OB, and equals $\frac{1}{2}\ OB$. § 189

Similarly, the middle point E of any other line AD may be proved to be at a distance from M equal to $\frac{1}{2}\ OB$.

Therefore, the required locus is the circumference of a ⊙ having the middle point of AO for centre, and $\frac{1}{2}\ OB$ for radius. Q. E. F.

Ex. 175. *A straight line moves so that it remains parallel to a given line, and touches at one end a given circumference. Find the locus of the other end.*

Let O be the centre of the given ⊙, AB the given line, CD the moving line that touches the given ⊙ at C and is ∥ to AB.



Proof. Draw OM ∥ to AB and equal to CD.

Draw OC, MD.

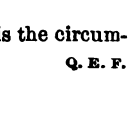
The figure $OCDM$ is a □. § 183

$\therefore MD = OC$. § 178

Therefore, the distance of D from M is constant, and the locus of D is a circumference equal to the given circumference and having the point M for centre. Q. E. F.

Ex. 176. *A straight rod moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its middle point.*

Let OA, OB be two fixed rods ⊥ to each other, AB the moving rod, M its middle point.



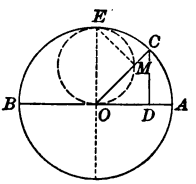
Proof. Draw OM.

Since $\angle AOB = 90°$, the point M is the centre of a ⊙ which passes through O, A, and B. Ex. 21

$\therefore OM = \frac{1}{2}\ AB$.

That is to say, OM is constant in length, and the locus is the circumference of a ⊙ having O for centre and $\frac{1}{2}\ AB$ for radius. Q. E. F.

Ex. 177. *In a given circle let AOB be a diameter, OC any radius, CD the perpendicular from C to AB. Upon OC take OM equal to CD. Find the locus of the point M as OC turns about O.*



Proof. Draw the radius $OE \perp$ to AB, and draw EM.

The $\triangle OEM$ and OCD are equal. § 143

For $OE = OC$, § 217

$OM = CD$, Const.

and $\angle EOM = \angle OCD$. § 110

$\therefore \angle OME = \angle OCD = 90°$. § 128

Therefore, the locus of the point M is the circumference of a $\odot$ described upon the radius OE as a diameter. Ex. 170

Q. E. F.

Ex. 178. *To construct an equilateral triangle, having given the radius of the circumscribed circle.*

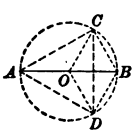
Describe a $\odot$ with the given radius.

Draw any diameter AOB.

With centre B and the given radius cut the $\odot$ at C and D.

Draw AC, AD, CD.

$\triangle ACD$ is the $\triangle$ required.



Proof. Draw OC, OD, BC, BD.

$\triangle OBC$ is equilateral by construction.

$\therefore \angle BOC = 60°$. § 146

$\therefore \angle COD = 120°$.

Now $\angle AOC = \angle AOD = 120°$, § 86

(being supplements of the $\measuredangle BOC$ and BOD).

$\therefore$ arc AC = arc AD = arc DC. § 236

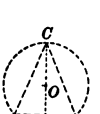
$\therefore AC = AD = DC$. § 241

Q. E. F.

Ex. 179. *To construct an isosceles triangle, having given the angle at the vertex and the base.*

Draw a line AB equal to the given base.

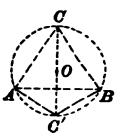
Construct upon AB a segment capable of containing the given angle. § 318



Draw the $\perp$ bisector of AB, and let it cut the arc at C. Draw AC, BC.

$\triangle ABC$ is the $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 180. *To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.*



Describe a ⊙ with the given radius.

Draw a chord AB equal to the given base.

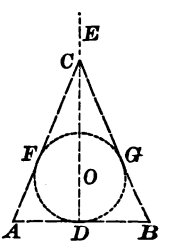
Draw the ⊥ bisector of AB, and let it meet the circle at C and C'.

Draw AC, AC', BC, BC'.

$\triangle ABC$, ABC' answer the given conditions, as is obvious from the construction and § 160. Q. E. F.

Discussion. If the given base is greater than twice the given radius, the problem is impossible.

Ex. 181. *To construct an isosceles triangle, having given the base and the radius of the inscribed circle.*



Draw a line AB equal to the given base.

Erect the ⊥ bisector DE.

On DE take DO equal to the given radius.

With centre O and radius DO describe a ⊙.

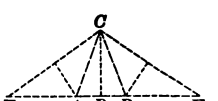
Draw tangents to the ⊙ from A and B, and let them intersect at C. § 317

$\triangle ABC$ is the $\triangle$ required.

Proof. Let AC meet the ⊙ at F, and BC meet it at G.

Then	$AF = AD$, and $BG = BD$.	§ 261
But	$AD = BD$.	Const.
	$\therefore AF = BG$.	Ax. 1
Also	$FC = GC$.	
Adding, we have	$AC = BC$.	Q. E. F.

Ex. 182. *To construct an isosceles triangle, having given the perimeter and the altitude.*



Draw EF equal to the given perimeter.

Bisect EF at D, and erect a ⊥ DC equal to the given altitude.

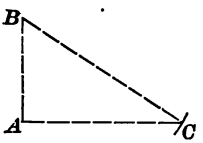
Draw EC, FC, and draw their ⊥ bisectors.

Let these bisectors cut EF at A and B.

Draw AC, BC. $\triangle ABC$ is the $\triangle$ required.

Proof.	$CE = CF$, $CA = EA$, $CB = FB$.	§ 160
	$\therefore CA + AB + BC = EA + AB + BF = EF$.	Ax. 2
Also	$\angle E = \angle F = \angle ACE = \angle BCF$.	§ 145
	$\therefore \triangle ACE = \triangle BCF$.	§ 139
	$\therefore AC = BC$.	§ 128
		Q. E. F.

Ex. 183. *To construct a right triangle, having given the hypotenuse and one leg.*

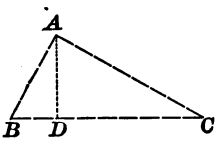


Construct a rt. $\angle BAC$.

Take AB equal to the given leg, and with centre B and radius equal to the given hypotenuse cut AC at C.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 184. *To construct a right triangle, having given one leg and the altitude upon the hypotenuse.*



Construct first a rt. $\triangle ABD$, having for hypotenuse the length AB equal to the given leg, and for one leg a length AD equal to the given altitude. Ex. 183

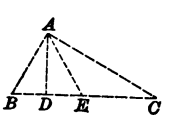
Draw $AC \perp$ to AB, and produce AC to meet BD produced at C.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 185. *To construct a right triangle, having given the median and the altitude drawn from the vertex of the right angle.*

Analysis. Let ABC be the $\triangle$ required. We know the altitude AD, and the median AE. Also we know that $BE = AE = CE$. Ex. 21

Construction. Construct the rt. $\triangle ADE$, in which are given the leg AD and the hypotenuse AE. Ex. 183



Produce DE to C, making $EC = AE$.

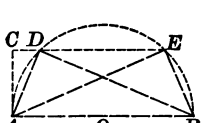
Draw AC, and construct at A a rt. $\angle$ whose side AB meets CD produced at B.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 186. *To construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.*

Upon a line AB equal to the given hypotenuse construct a semicircle. Erect at A a $\perp AC$ equal to the given altitude.

Through C draw a line $\parallel$ to AB, cutting the semicircle at D and E.



Draw DA, DB, EA, EB.

The $\triangleq ADB$, AEB are the $\triangleq$ required.

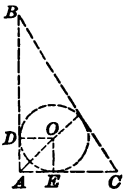
Proof. The $\measuredangle D$ and E are rt. $\measuredangle$ (§ 290), and the altitudes of the $\triangleq$ are equal each to AC. § 181

Q. E. F.

Discussion. Since the $\triangleq ADB$, AEB are equal, there is really only one solution. If $AC = \frac{1}{2}AB$, CE will be a tangent and the $\triangle$ will be isosceles; if $AC > \frac{1}{2}AB$, the problem is impossible.

Ex. 187. *To construct a right triangle, having given the radius of the inscribed circle and one leg.*

Analysis. Let ABC be the $\triangle$ required, with the rt. $\angle$ at A, and O the centre of the inscribed $\odot$. We have given AB and OD.



We know that O lies in the bisector of $\angle A$ (§ 315), that $\measuredangle D$ and E are rt. $\measuredangle$ (§ 254), and hence we infer that the figure $ADOE$ is a square.

Construction. Construct a rt. $\angle BAC$, and upon one side lay off a length AB equal to the given leg.

Draw the bisector AO of the $\angle A$.

Upon AB take AD equal to the given radius, and at D erect a $\perp$ cutting AO at O.

With centre O and radius OD describe a $\odot$.

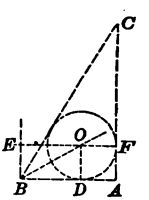
Draw from B to this $\odot$ a tangent meeting AC at C.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 188. *To construct a right triangle, having given the radius of the inscribed circle and an acute angle.*

Analysis. Suppose the problem solved, ABC the required $\triangle$, $\angle A$ the rt. $\angle$, and O the centre of the inscribed $\odot$.

The given parts are $\angle B$ and the radius OD. We know that O lies in the bisector of $\angle B$ (§ 315), and also in a line EO drawn $\parallel$ to BA and at the distance OD from BA. We also know that if EO is produced it will pass through the point of contact of the side AC. § 256

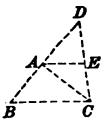


Construction. Construct an $\angle ABC$ equal to the given $\angle$. Draw the bisector BO of the $\angle B$. Erect at B a $\perp BE$ equal to the given radius. Draw through E a line $\parallel$ to BA, meeting BO at O. With centre O and radius BE describe a $\odot$. Produce EO to meet the $\odot$ at F. Draw through F a tangent to the $\odot$, meeting the sides of the $\angle B$ at A and C.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 189. *To construct a right triangle, having given an acute angle and the sum of the legs.*

Analysis. Let ABC be the required $\triangle$, and $\angle A$ the rt. $\angle$. The given parts are the $\angle B$ and the length $AB + AC$. Produce BA to D, making AD equal to AC, and draw DC.



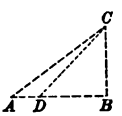
Since $\triangle ACD$ is a rt. isosceles $\triangle$, $\angle D = 45°$.

We can construct $\triangle BDC$, since we know BD and the $\measuredangle B$ and D (§ 310); then a $\perp$ erected at the middle point of DC will pass through A. Ex. 35

Construction. Construct a $\triangle BDC$ from the known parts. § 310
Draw the $\perp$ bisector of DC, meeting BD at A, and draw AC.
$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 190. *To construct a right triangle, having given an acute angle and the difference of the legs.*

Analysis. Let ABC be the $\triangle$ required, AC the hypotenuse. Upon the greater leg AB lay off a length AD equal to the given difference of the legs. We know the $\angle A$ and AD.

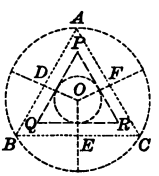


Since $BC = BD$, $\angle BDC = \angle DCB = 45°$. § 145
Hence, by constructing at D an $\angle BDC$ equal to 45°, the point C will be determined.

Construction. Construct an $\angle BAC$ equal to the given angle.
Upon AB take AD equal to the given difference of the legs.
Construct the $\angle BDC$ equal to 45°, and drop from C the $\perp CB$.
$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 191. *To construct an equilateral triangle, having given the radius of the inscribed circle.*

With the given radius and centre O describe a $\odot$; then describe a larger concentric $\odot$, and in this last circle inscribe an equilateral $\triangle ABC$. Ex. 178



Drop from O to the sides AB, AC, BC the $\perp$s OD, OF, OE, respectively.

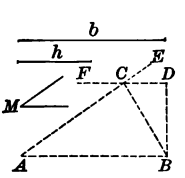
Through the points where these $\perp$s cut the smaller $\odot$ draw tangents intersecting in P, Q, R.

Then $\triangle PQR$ is the $\triangle$ required.

Proof. The tangent PQ is $\perp$ to OD, § 254
and therefore $\parallel$ to AB. § 104
Similarly, PR is $\parallel$ to AC, and $QR \parallel$ to BC.
$\therefore \angle QPR = \angle BAC = 60°$. § 176
Similarly, $\angle PQR = \angle PRQ = 60°$. $\therefore \triangle PQR$ is equilateral. § 148
Q. E. F.

Ex. 192. *To construct a triangle, having given the base, the altitude, and an angle at the base.*

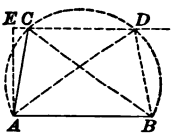
Draw a line AB equal to the given base.
At A construct an $\angle BAE$ equal to the given angle.
Erect at B a $\perp BD$ equal to the given altitude.
Draw through D a line $DF \parallel$ to BA.
Let this parallel cut AE at C, and draw BC.
$\triangle ABC$ is the triangle required.



Proof. $\triangle ABC$ has the parts AB and $\angle BAC$ by direct construction, and its altitude equals the given altitude by construction and § 181. Q. E. F.

Ex. 193. *To construct a triangle, having given the base, the altitude, and the angle at the vertex.*

Analysis. If AB is the given base, one locus for the vertex is an arc forming with AB a segment capable of containing the given angle (§ 293). Another locus for the vertex is a parallel to AB, drawn at a distance from AB equal to the given altitude.



Construction. Upon a line AB equal to the given base, construct a segment capable of containing the given $\angle$ at the vertex. § 318

Erect a $\perp$ AE equal to the given altitude.

Draw through E a line $\parallel$ to AB, cutting the arc at C and D.

Draw AC, BC, AD, BD.

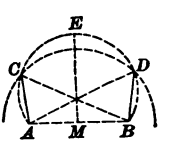
The $\triangle$ ABC, ABD both satisfy the given conditions, as is evident from the construction and from § 181. Q. E. F.

Discussion. Since the $\triangle$ ABC, ABD are evidently equal, if AE equals the height of the segment there is but one $\triangle$, and the $\triangle$ is isosceles. If AE is greater than the height of the segment, the problem is impossible.

Ex. 194. *To construct a triangle, having given the base, the corresponding median, and the angle at the vertex.*

Analysis. If AB is the given base, one locus for the vertex is an arc forming with AB a segment capable of containing the given $\angle$. § 293

Another locus for the vertex is a circumference described with the middle point of AB for centre, and the given length of the median for radius.



Construction. Upon AB construct a segment capable of containing the given $\angle$ at the vertex. Bisect AB at M.

With centre M and radius equal to the length of the given median, describe a $\odot$ cutting the arc of the segment at C and D.

Draw AC, BC, AD, BD.

The $\triangle$ ABC, ABD satisfy the given conditions. Q. E. F.

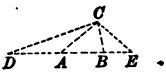
Discussion. The two $\triangle$ ABC, ABD are equal and form but one solution. In order that the problem may be possible, the given median must have a value between half the given base and the height ME of the segment constructed upon the base.

Ex. 195. *To construct a triangle, having given the perimeter and the angles.*

Analysis. Let ABC be the $\triangle$ required. Upon AB produced take AD equal to AC, and BE equal to BC.

Then DE is equal to the given perimeter.

Since $\triangle ACD$, BCE are isosceles, and $\measuredangle A$ and B of $\triangle ABC$ are exterior $\measuredangle$, it is easily seen that $\angle ADC = \frac{1}{2}\angle BAC$, and $\angle BEC = \frac{1}{2}\angle ABC$. §§ 145, 137



Construction. Draw DE equal to the given perimeter.

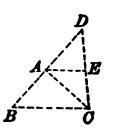
Construct at D an $\angle$ equal to $\frac{1}{2}\angle A$, and at E an $\angle$ equal to $\frac{1}{2}\angle B$; this determines the point C.

Draw through C a line CA, making $\angle ACD$ equal to $\frac{1}{2}\angle A$, and a line CB, making $\angle BCE$ equal to $\frac{1}{2}\angle B$.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 196. *To construct a triangle, having given one side, an adjacent angle, and the sum of the other sides.*

Analysis. Suppose the problem solved, and ABC to be the $\triangle$ required. We know BC, the $\angle B$, and the sum $BA + AC$. Produce BA, making AD equal to AC, and draw DC. $\triangle ADC$ is isosceles. We now see that $\triangle BDC$ is determined; for $\angle B$ and BC are given, and we can make $BD = BA + AC$. After $\triangle BDC$ has been constructed, the point A may be found by erecting a $\perp$ at the middle point of DC.



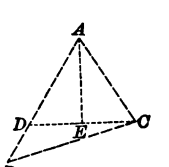
Construction. Construct $\triangle BDC$. § 309

Draw the $\perp$ bisector of DC, cutting BD at A, and draw AC.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 197. *To construct a triangle, having given one side, an adjacent angle, and the difference of the other sides.*

Analysis. Suppose the problem solved, and let ABC be the $\triangle$ required. We know BC, the $\angle B$, and the difference $AB - AC$. If, then, we take upon BA a length BD equal to $AB - AC$, we obtain a $\triangle BDC$, which we can construct from the given parts; then a $\perp$ erected at the middle point of DC will meet BD produced, and determine the point A.



Construction. Construct $\triangle BDC$. § 309

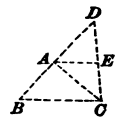
Draw the $\perp$ bisector of DC, meeting BD produced at A.

Draw AC.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 198. *To construct a triangle, having given the sum of two sides and the angles.*

Analysis. Suppose the problem solved and let ABC be the $\triangle$ required. We know the $\measuredangle$ and the sum $AB + AC$. Produce BA, making AD equal to AC. The $\triangle ADC$ is isosceles.



$$\therefore \angle D = \tfrac{1}{2} \angle BAC. \qquad \text{§§ 137, 145}$$

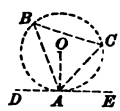
Since also $\angle B$ and BD are known, $\triangle BDC$ is determined.

Construction. Construct $\triangle BDC$. § 310

Draw the $\perp$ bisector of DC, meeting BD at A, and draw AC.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 199. *To construct a triangle, having given one side, an adjacent angle, and the radius of the circumscribed circle.*



With the given radius describe a $\odot$.

In this $\odot$ draw a chord AB equal to the given side.

Through A draw another chord AC, making the $\angle BAC$ equal to the given $\angle$, and draw BC.

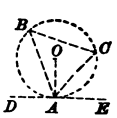
$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 200. *To construct a triangle, having given the angles and the radius of the circumscribed circle.*

With the given radius describe a $\odot$.

Through any point A of the circumference draw a tangent DE.

Through A draw the chords AB, AC, making the $\measuredangle DAB$, EAC equal, respectively, to two of the given $\measuredangle$. Draw BC.



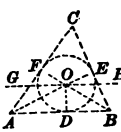
$\triangle ABC$ is the $\triangle$ required.

Proof. $\angle ABC = \angle EAC$, because each $\angle$ is measured by half the arc AC. §§ 289, 295

Similarly, $\angle ACB = \angle DAB$. Q. E. F.

Ex. 201. *To construct a triangle, having given the angles and the radius of the inscribed circle.*

Analysis. Let ABC be the $\triangle$ required, OD the radius of the inscribed $\odot$, meeting AB at D. We know the $\measuredangle A$, B, C, and the length OD. We also know that O is the intersection of the bisectors of the $\measuredangle A$, B, C; also that its distance from AB is equal to OD. Finally, we see that if GH is a line drawn through $O \parallel$ to AB, then $\angle BOH = \angle OBD = \tfrac{1}{2} \angle ABC$.



Construction. Construct the $\angle A$ and its bisector.

Draw $GH \parallel$ to AB at the given distance OD from AB, and let GH meet the bisector at O.

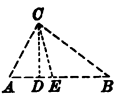
Draw OB, meeting AB at B, and making $\angle BOH$ equal to one half the given $\angle B$.

Draw BC, meeting AC at C, and making $\angle OBC$ equal to one half the given $\angle B$.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 202. *To construct a triangle, having given an angle, and the bisector and the altitude drawn from the vertex of the given angle.*

Analysis. Let ABC be the $\triangle$ required. There are given $\angle C$, the altitude CD, and the bisector CE. Therefore, the rt. $\triangle CDE$ is determined, and also the $\measuredangle ECA$, ECB, each being one half of the given $\angle C$.



Ex. 183

Construction. Construct $\triangle CDE$.

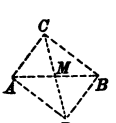
Produce DE in both ways.

Draw CA and CB, meeting DE produced at A and B, respectively, and making the $\measuredangle ECA$ and ECB each equal to one half the given $\angle C$.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 203. *To construct a triangle, having given two sides and the median corresponding to the third side.*

Analysis. Let ABC be the $\triangle$ required. We know AC, BC, and the median CM. Produce CM, making MD equal to CM, and draw DA, DB. The $\measuredangle AMD$, BMC are equal (§ 143); hence $AD = BC$. Therefore, the $\triangle ACD$ is determined, because its sides are all known; and when it has been constructed, the point B is found by producing AM so that $MB = AM$.



Construction. Construct $\triangle ACD$ (§ 312), draw the median AM, and produce it to B, making MB equal to AM. Draw BC.

$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 204. *To construct a triangle, having given the three medians.*

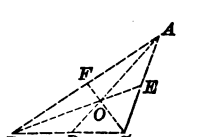
Analysis. Let ABC be the $\triangle$ required, AD, BE, CF the medians, O their common point of intersection.

Ex. 27

We know that the distance from O to each vertex is two thirds the length of the corresponding median. Therefore, the $\triangle BOC$ is determined, since we know OB, OC, and OD.

Ex. 203

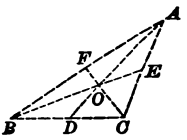
By producing BO by a length OE equal to $\frac{1}{2}BO$, and CO by a length OF equal to $\frac{1}{2}OC$, we obtain the points E and F. Finally, the lines BF, CE produced will meet at the point A.



Construction. Construct the $\triangle BOC$. Ex. 203

Produce BO, making OE equal to $\frac{1}{2}BO$, and produce CO, making OF equal to $\frac{1}{2}OC$.

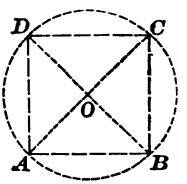
Draw BF and CE, and produce them to meet at A.



$\triangle ABC$ is the $\triangle$ required. Q. E. F.

Ex. 205. *To construct a square, having given the diagonal.*

Analysis. Suppose the problem solved, $ABCD$ the square, AC the diagonal. We know AC. We also know that the $\measuredangle B$ and D are rt. $\measuredangle$. Therefore, B and D must lie in the circumference of a $\odot$ having AC for diameter. Also, since B and D are each equidistant from A and C, B and D lie in the $\perp$ bisector of AC. § 160



Construction. Bisect AC at O.

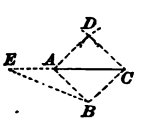
With centre O and radius OA describe a $\odot$, intersecting the $\perp$ bisector of AC at B and D.

Draw AB, BC, CD, DA.

$ABCD$ is the required square. Q. E. F.

Ex. 206. *To construct a square, having given the sum of the diagonal and one side.*

Analysis. Let $ABCD$ be the square required, AC the diagonal. Produce CA to E, making AE equal to AB, and draw BE.



$\measuredangle ABC$ and ABE are isosceles, and

$$\angle ACB = \angle CAB = 45°;$$

whence we can find $\angle AEB$ (§ 137); this determines $\triangle EBC$; then D can be found as in Ex. 205.

Construction. Draw a line EC equal to the given sum of AB and AC. Draw CB, making $\angle BCE$ equal to 45°, and EB, making $\angle BEC$ equal to $22\frac{1}{2}°$, meeting CB at B.

Erect $BA \perp$ to BC.

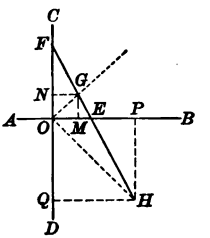
With centres A and C, and radius AB, describe arcs intersecting at D.

Draw DA, DC.

$ABCD$ is the square required. Q. E. F.

Ex. 207. *Given two perpendiculars, AB and CD, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF.* (*Two solutions.*)

Bisect the $\measuredangle$ at O, and let one bisector meet EF at G, and the other meet FE produced at H. Through G draw $GM \parallel$ to CD, and $GN \parallel$ to BA. Through H draw $HP \parallel$ to DC, and $HQ \parallel$ to BA. The figures $OMGN$, $OPHQ$ are squares satisfying the given conditions.



Proof. In the figure $OMGN$,

$\angle O = 90°$, Hyp.

$\angle M = \angle N = 90°$, § 107

and $\angle MGN = \angle MON$. § 176

$\therefore \angle MGN = 90°$.

Also $GN = GM$, § 162

and $GN = OM$.

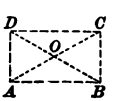
Now $GM = ON$. § 180

$\therefore$ the figure $OMGN$ is a square. § 168

Similarly, the figure $OPHQ$ is a square. Q. E. F.

Ex. 208. *To construct a rectangle, having given one side and the angle between the diagonals.*

Analysis. Let $ABCD$ be the rectangle required, AB the given side, $\angle AOB$ the given $\angle$. Since the diagonals of a rectangle are equal (Ex. 69), and bisect each other (§ 184), we know that $AO = BO$; hence, the isosceles $\triangle AOB$ is determined, and may be constructed. The points C and D are then found by properly producing AO and BO.



Construction. Construct the $\triangle AOB$. Ex. 179

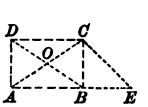
Produce AO, making OC equal to AO, and BO, making OD equal to BO.

Draw BC, CD, DA. $ABCD$ is the rectangle required. Ex. 68

Q. E. F.

Ex. 209. *To construct a rectangle, having given the perimeter and the diagonal.*

Analysis. Let $ABCD$ be the rectangle required. In the rt. $\triangle ABC$ we know AC and $AB + BC$. To construct this $\triangle$, prolong AB to E, making BE equal to BC. At E make an $\angle$ of 45°, and let its side meet at C the $\odot$ described from A as centre with AC as radius. From C drop the $\perp CB$.



Construction. Construct the rt. $\triangle ABC$.

Bisect AC at O.

Draw BO, and produce it to D, making OD equal to BO.

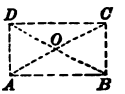
Draw DC and DA.

$ABCD$ is the rectangle required. Ex. 68

Q. E. F.

Ex. 210. *To construct a rectangle, having given the perimeter and the angle between the diagonals.*

Analysis. Let $ABCD$ be the rectangle required, $\angle AOB$ the given $\angle$. Since $\triangle OAB$ is isosceles, the $\angle OAB$ may be easily found, and then its complement, the $\angle ACB$. Hence, in the rt. $\triangle ABC$, we know the $\measuredangle$ and the sum $AB + BC$ (which is equal to half the given perimeter). Therefore, we construct the $\triangle ABC$, and then proceed as in Ex. 209.



Construction. Make $\angle OAB$ equal to $\frac{1}{2}(180° - \angle AOB)$.

Construct the $\triangle ABC$. Ex. 198

Bisect AC at O.

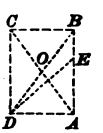
Draw BO and produce it to D, making OD equal to OB.

Draw DC, DA. $ABCD$ is the rectangle required. Ex. 68

Q. E. F.

Ex. 211. *To construct a rectangle, having given the difference of two adjacent sides and the angle between the diagonals.*

Analysis. Let $ABCD$ be the rectangle required, and E a point in AB, such that $AE = AD$. We know EB, the $\angle AOB$, and we know that $\angle B = \frac{1}{2}(180° - \angle AOB)$.



In $\triangle ADE$, $AE = AD$.

$\therefore \angle AED = \angle ADE$;

and since the $\triangle$ is a rt. $\triangle$,

$\angle AED = 45°$, and $\angle BED = 180° - 45° = 135°$.

Therefore, in $\triangle BDE$, a side and the adjacent $\measuredangle$ are known. After this $\triangle$ has been constructed, drop a $\perp$ from D to meet BE produced, etc.

Construction. Construct the $\triangle BDE$. § 310

From D draw $DA \perp$ to BE, meeting BE produced at A.

Bisect BD at O.

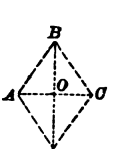
Draw AO, and produce it, making OC equal to AO. Draw CB, CD.

$ABCD$ is the rectangle required. Ex. 68

Q. E. F.

Ex. 212. *To construct a rhombus, having given the two diagonals.*

Analysis. The diagonals of a rhombus bisect each other (§ 184), and are $\perp$ to each other (Ex. 71); hence, the construction is obvious.



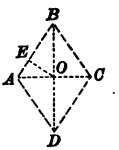
Construction. Draw AC equal to one diagonal.

Draw the $\perp$ bisector of AC, cutting AC at O; upon it lay off in opposite directions from C the lengths OB, OD, each equal to half the other diagonal.

$ABCD$ is the rhombus required. Q. E. F.

Ex. 213. *To construct a rhombus, having given one side and the radius of the inscribed circle.*

Analysis. Since the diagonals of a rhombus bisect each other (Ex. 71), their intersection O is a point equidistant from the sides (§ 162), and therefore the centre of the inscribed ⊙. Hence, if we draw $OE \perp$ to AB, OE will be equal to the radius of the inscribed ⊙. Now in the rt. $\triangle ABO$ we know the hypotenuse AB and the altitude OE. This $\triangle$ once constructed, the rest is obvious.



Construction. Construct the $\triangle AOB$. Ex. 186

Produce AO, making OC equal to AO, and BO, making OD equal to BO.

Draw AD, DC, CB. $ABCD$ is the rhombus required. Q. E. F.

Ex. 214. *To construct a rhombus, having given one angle and the radius of the inscribed circle.*

Analysis. Let $ABCD$ be the rhombus required, $\angle ABC$ the given $\angle$, O the centre of the inscribed ⊙. Draw radii to the points of contact E, F, G, H.

The $\angle FOG = 180° - \angle GCF$, §§ 206, 254

and $\angle ABC = 180° - \angle GCF$. § 115

$\therefore \angle FOG = \angle ABC$. Ax. 1

Since also $\angle EOF = 180° - \angle ABC$,

GOE is a straight line. § 90

Similarly, FOH is a straight line.

Construction. With centre O and the given radius, describe a ⊙.

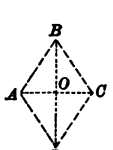
Draw through O two diameters GOE, FOH, making with each other the $\angle GOF$ equal to the given $\angle$.

Through the points E, F, G, H draw tangents intersecting in A, B, C, D.

$ABCD$ is the required rhombus. Q. E. F.

Ex. 215. *To construct a rhombus, having given one angle and one of the diagonals.*

Analysis. Let $ABCD$ be the rhombus required, $\angle ABC$ the given $\angle$, AC the given diagonal. The $\angle BAD = 180° - \angle ABC$ (§ 115), and is therefore known. Also the diagonals of a rhombus bisect each other at rt. $\measuredangle$, and bisect the $\measuredangle$ of the rhombus. Ex. 71



Construction. Draw AC.

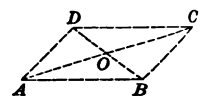
Through A draw AB and AD, each making an $\angle$ with AC equal to $\frac{1}{2}(180° - \angle ABC)$.

Draw the $\perp$ bisector of AC, meeting AB at B and AD at D.

Draw CB, CD. The figure $ABCD$ is the rhombus required. Q. E. F.

Ex. 216. *To construct a rhomboid, having given one side and the two diagonals.*

Analysis. Let $ABCD$ be the rhomboid required, AB the given side, O the intersection of the diagonals. Since the diagonals bisect each other (§ 184) we know the three sides of the $\triangle AOB$; then C and D are found by producing AO and BO.



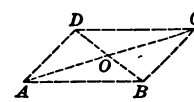
Construction. Construct the $\triangle AOB$. § 312

Produce AO, making OC equal to AO, and BO, making OD equal to BO. Draw BC, AD, DC.

$ABCD$ is the rhomboid required. Q. E. F.

Ex. 217. *To construct a rhomboid, having given the diagonals and the angle between them.*

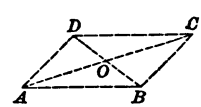
Analysis. Let $ABCD$ be the rhomboid required, $\angle AOB$ the given $\angle$, O the intersection of the diagonals. In the $\triangle AOB$ we know two sides OA, OB, and the included $\angle$. § 184



Construction. Construct the $\triangle AOB$, and then proceed as in Ex. 216. Q. E. F.

Ex. 218. *To construct a rhomboid, having given one side, one angle, and one diagonal.*

Analysis. Let $ABCD$ be the rhomboid required, $\angle A$ the given $\angle$, AB the given side, AC the given diagonal. The $\angle ABC = 180° - \angle BAD$. Hence, the $\triangle ABC$ is determined.



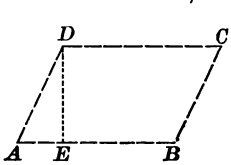
Construction. Construct the $\triangle ABC$. § 311

Through A draw $AD \parallel$ to BC, and through C draw $CD \parallel$ to BA.

$ABCD$ is the rhomboid required. Q. E. F.

Ex. 219. *To construct a rhomboid, having given the base, the altitude, and one angle.*

Analysis. Let $ABCD$ be the rhomboid required. We know AB, the $\angle A$, and the altitude DE. Therefore, the $\angle ADE$ is known, and the rt. $\triangle ADE$ is determined.



Construction. Construct the rt. $\triangle ADE$. § 310

Produce AE, making AB equal to the given base. Through B draw $BC \parallel$ to AD, and through D draw $DC \parallel$ to AB.

The figure $ABCD$ is the rhomboid required. Q. E. F.

Ex. 220. *To construct an isosceles trapezoid, having given the bases and one angle.*

Analysis. Let $ABCD$ be the trapezoid required, $\angle A$ the given $\angle$.

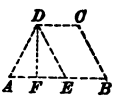
Draw $DE \parallel$ to CB.

$DE = BC$. § 180

But $BC = AD$.

$\therefore DE = AD$. Ax. 1

$\therefore \angle DEA = \angle A$. § 145



Also, since $AE = AB - EB$ and $EB = DC$, $AE = AB - DC$, and is therefore known. $\therefore$ the $\triangle ADE$ is determined. This $\triangle$ constructed, the rest of the construction is obvious.

Construction. Construct the $\triangle ADE$. § 310

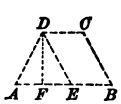
Produce AE, making AB equal to the longer base.

Draw DC equal to the shorter base and $\parallel$ to AB, and draw BC.

The figure $ABCD$ is the isosceles trapezoid required. Q. E. F.

Ex. 221. *To construct an isosceles trapezoid, having given the bases and the altitude.*

Analysis. Let $ABCD$ be the trapezoid required, DF the altitude. Draw $DE \parallel$ to CB. The $\triangle ADE$ is isosceles (see last Analysis), and $AE = AB - DC$. Also DF bisects AE. § 149



Construction. Draw AB equal to the longer base.

Upon AB lay off AE equal to $AB - DC$.

Draw the $\perp$ bisector to AE and lay off upon it a length FD equal to the given altitude. Draw AD.

Through B draw $BC \parallel$ to ED, and through D draw $DC \parallel$ to AB.

The figure $ABCD$ is the isosceles trapezoid required. Q. E. F.

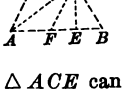
Ex. 222. *To construct an isosceles trapezoid, having given the bases and the diagonal.*

Analysis. Let $ABCD$ be the trapezoid required, AC the given diagonal. Draw the altitude CE, and also draw $CF \parallel$ to DA.

Then $AF = DC$, § 180

and $FB = AB - DC$.

Also, since the $\triangle BCF$ is isosceles (see Analysis to Ex. 220), $FE = \frac{1}{2} FB$. § 149



$\therefore AE = DC + \frac{1}{2}(AB - DC)$, and is known. $\therefore$ the rt. $\triangle ACE$ can be constructed. When this has been done, the rest is obvious.

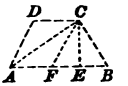
Construction. Construct the rt. $\triangle ACE$. Ex. 183

Produce AE, making AB equal to the longer base.

Draw $DC \parallel$ to AB and equal to the shorter base.

Draw BC and AD.

The figure $ABCD$ is the isosceles trapezoid required. Q. E. F.

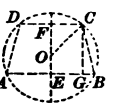


Ex. 223. *To construct an isosceles trapezoid, having given the bases and the radius of the circumscribed circle.*

Analysis. Let $ABCD$ be the trapezoid required, O the centre of the circumscribed $\odot$. A diameter $\perp$ to AB is $\perp$ to CD (§ 107), and bisects both AB and CD. § 245

Let this diameter meet AB at E, CD at F. Draw $CG \parallel$ to FE, meeting AB at G; then $EG = FC = \frac{1}{2} DC$.

Hence, we have the following construction:



Construction. With any centre O and the given radius describe a $\odot$.

Draw in this $\odot$ a chord AB equal to the longer base.

Draw a diameter $\perp$ to AB, meeting AB at E.

Upon AB lay off EG equal to half the shorter base.

Draw $GC \parallel$ to EO, meeting the $\odot$ at C.

Draw $CD \parallel$ to BA, meeting the $\odot$ again at D.

Draw BC, AD.

$ABCD$ is the isosceles trapezoid required. Q. E. F.

Ex. 224. *To construct a trapezoid, having given the four sides.*

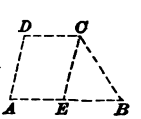
Analysis. Let $ABCD$ be the trapezoid required. Draw $CE \parallel$ to DA. $\triangle CBE$ is determined, because BC is given, $CE = AD$ (§ 180), and $EB = AB - CD$.

Construction. Construct $\triangle BCE$. § 312

Produce BE to A, making AB equal to the longer base.

Draw $AD \parallel$ to EC and equal to EC, and draw DC.

$ABCD$ is the required trapezoid. Q. E. F.



Ex. 225. *To construct a trapezoid, having given the two bases and the two diagonals.*

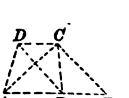
Analysis. Let $ABCD$ be the trapezoid required. Draw $CE \parallel$ to the diagonal DB, and produce AB to meet CE at E. $\triangle AEC$ is determined, since AC is given, $CE = BD$ (which is given), and $AE = AB + DC$.

Construction. Construct $\triangle AEC$. § 312

Through B draw $BD \parallel$ and equal to EC.

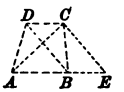
Draw AD, DC, BC.

$ABCD$ is the required trapezoid. Q. E. F.



Ex. 226. *To construct a trapezoid, having given the bases, one diagonal, and the angle between the diagonals.*

Analysis. Let AB, DC be the given bases, AC the given diagonal. The $\angle ACE$ equals the angle of the diagonals. If this angle is acute, and AE is greater than, or equal to, AC, the $\triangle ACE$ is determined. § 311



Construction. Construct $\triangle AEC$.

Upon AE take AB equal to one of the bases.

Draw $CD \parallel$ to EA and equal to the other base.

Draw BC and AD.

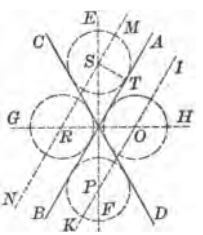
The figure $ABCD$ is the trapezoid required. Q. E. F.

Discussion. For other cases, see the discussion of the problem in § 311.

Ex. 227. *To construct a circle which has the radius r and which also touches each of two intersecting lines AB and CD.*

Analysis. One locus of the centre consists of the bisectors of the $\measuredangle$ formed by the lines AB and CD. Ex. 155

Another locus consists of the two parallels to either AB or CD, drawn at a distance from it equal to the given radius.



Construction. Draw the bisectors EF, GH of the $\measuredangle$ formed by the given lines AB, CD.

Draw IK and $MN \parallel$ to AB, and at a distance from AB equal to the given radius r.

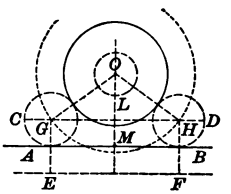
Let the parallels cut the bisectors at the points O, P, R, S.

With these points as centres and r as radius, describe Ⓢ.

Each ⊙ will satisfy the given conditions. Q. E. F.

Ex. 228. *To construct a circle which has the radius r and which also touches a given line AB and a given circle K.*

Analysis. Let O be the centre of the given ⊙, R its radius. One locus of the centre of the required ⊙ consists of the parallels to AB drawn at the distance r from AB. Another locus consists of the circumferences described about O as centre with the radii $R + r$ and $R - r$. The points in which the two parallels intersect the two circumferences are the centres of Ⓢ which will satisfy the given conditions.



The construction of two of these Ⓢ is sufficiently evident from the figure. Q. E. F.

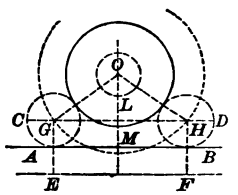
Discussion. Let d denote the distance OM from O to AB.

(1) If $d < R + 2r$, the problem is impossible.

(2) If $d = R + 2r$, there is *one* solution, and the centre of the ⊙ lies in the line OM.

(3) If d has any value between $R + 2r$ and R, there are *two* solutions, as shown in the figure.

(4) If $d = R$, CD will pass through the point L and determine three centres, one at L, the others on the ⊙ OG. But the other parallel EF will now be tangent to the ⊙ OG and determine by the point of contact a fourth centre. Therefore, in this case there are *four* solutions.



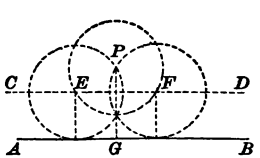
(5) If $d < R$ but $> R - 2r$, the parallel CD will cut both auxiliary Ⓢ and determine four centres; and the parallel EF will cut the ⊙ OG and determine two centres. Therefore, there will be *six* solutions.

(6) If $d = R - 2r$ (R being supposed $> 2r$), EF will pass through L and determine three centres; CD will determine four more. Therefore, there will be *seven* solutions.

(7) If $d < R - 2r$, CD and EF will cut each auxiliary ⊙ twice; hence, we shall have *eight* solutions.

Ex. 229. *To construct a circle which has the radius r, and which also passes through a given point P and touches a given line AB.*

Analysis. The centre of the required ⊙ must lie in the circumference described from P as centre with r as radius. It must also lie in a parallel to AB drawn at the distance r from AB on the *same* side of AB as the point P. The points common to this circumference and this parallel are the centres of Ⓢ which if described with the radius r will satisfy the given conditions.

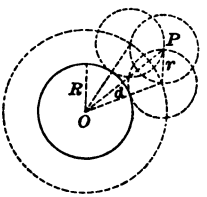


Q. E. F.

Discussion. It is obvious that the parallel to AB at the distance r on *the other side* of AB is not in general a locus for the centre of the required ⊙. Let d denote the distance PG from P to AB. If $d > 2r$, the problem is impossible. If $d = 2r$, CD is tangent to the auxiliary ⊙, and there is *one* solution. If $d < 2r$, there are *two* solutions, as shown in the figure. If $d = 0$, P lies in AB, and there are two Ⓢ, each touching AB at P, the centre of one lying in the parallel CD, the centre of the other lying in the corresponding parallel on the other side of AB.

Ex. 230. *To construct a circle which has the radius r, and which also passes through a given point P and touches a given circle K.*

Analysis. Let R denote the radius of the given ⊙, and O denote its centre. One locus of the required centre is a circumference described from P as centre with radius r. Further, since the required ⊙ must touch the given ⊙, and the distance between the centres of two Ⓢ which touch each other is equal to the sum or the difference of their radii (Ex. 92), it follows that a second locus of the required centre is a circumference described from O as centre with a radius equal to $R + r$ or $R - r$. The points common to these loci are the centres of the circles which satisfy the given conditions.



Q. E. F.

Discussion. Let d denote the distance from P to O. In order that the ⊙ with radius $R + r$ may cut the auxiliary ⊙ described about P, it is necessary that

$$d < (R + r) + r, \text{ or } d < R + 2r,$$

and

$$d > (R + r) - r, \text{ or } d > R.$$

The second condition requires that P should lie without the given ⊙. Then if $d < R + 2r$, there are *two* solutions; if $d = R + 2r$, *one* solution; if $d > R + 2r$, *no* solution.

In order that the ⊙ with radius $R - r$ may cut the auxiliary ⊙ described about P, it is necessary that

$$d < (R - r) + r, \text{ or } d < R,$$

and

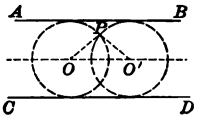
$$d > (R - r) - r, \text{ or } d > R - 2r.$$

The first condition requires P to be within the given ⊙. Then, if $d > R - 2r$, there are *two* solutions; if $d = R - 2r$, *one* solution; if $d < R - 2r$, *no* solution.

If $d = R$, P lies in the given circumference, and two Ⓢ can be described with radius r, both touching the given ⊙ at P, one externally, the other internally.

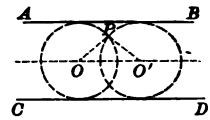
Ex. 231. *To construct a circle which shall touch two given parallels and pass through a given point P.*

Analysis. Let AB and CD be the given parallels. One locus of the required centre is the parallel drawn midway between AB and CD. Ex. 154



The radius of the required ⊙ is equal to half the distance between AB and CD. Therefore, another locus for the centre is a circumference

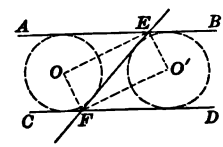
described from P as centre with half the distance between AB and CD for radius. Q. E. F.



Discussion. If P lies between AB and CD, there are *two* solutions. If P is on AB or CD, there is *one* solution. If P is outside the parallels, the problem is impossible.

Ex. 232. *To construct a circle which shall touch three given lines, two of which are parallel.*

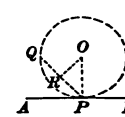
Analysis. Let AB and CD be the parallel lines, and let the third line cut AB at E, and CD at F. The centres of ⓢ that will satisfy the conditions are the intersections of the bisectors of the $\measuredangle$ formed at the points E and F (Ex. 155), and must also be situated between AB and CD. Ex. 154 Q. E. F.



Discussion. There are *two* solutions, one on each side of EF.

Ex. 233. *To construct a circle which shall touch a given line AB at P, and pass through a given point Q.*

Analysis. One locus for the centre is a $\perp$ to AB erected at P (Ex. 153), and another locus is the $\perp$ bisector of the line joining P and Q. Ex. 152 Q. E. F.

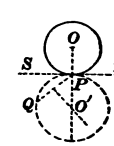


Discussion. If Q is in AB, the problem is impossible.

Ex. 234. *To construct a circle which shall touch a given circle at P and pass through a given point Q.*

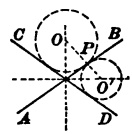
Analysis. One locus for the centre is the $\perp$ bisector of the line joining P and Q. Ex. 152

Another locus is the straight line passing through P and the centre of the given ⊙. §§ 221, 265 Q. E. F.



Discussion. Since two straight lines intersect in only one point, there cannot be more than *one* solution. Let ST be the tangent to the given ⊙ drawn through the point P. If Q and the given ⊙ lie on opposite sides of ST, the required ⊙ touches the given ⊙ externally. If Q and the given circle lie on the same side of ST, there is internal contact; and the required ⊙ contains the given ⊙, or the given ⊙ contains the required ⊙, according as Q lies without or within the given ⊙. If Q lies on ST, the problem is impossible.

Ex. 235. *To construct a circle which shall touch two given lines and touch one of them at a given point P.*

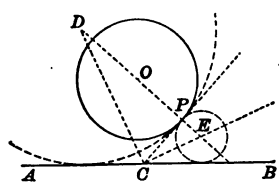


Analysis. Let AB, CD be the given lines, and let P be a point in AB. The centre of the required ⊙ must lie in one of the bisectors of the ∠ formed by AB and CD; and also in a ⊥ to AB erected at P. Q. E. F.

Discussion. There are in general *two* Ⓢ satisfying the given conditions. If P coincides with the intersection of AB and CD, the problem is impossible. If AB is ∥ to CD, there is only *one* ⊙, and its radius is equal to half the distance between AB and CD.

Ex. 236. *To construct a circle which shall touch a given line and touch a given circle at a point P.*

Analysis. Draw a tangent to the given ⊙ at the point P; then the required ⊙ must touch this tangent at P (§ 221). This reduces the problem to Ex. 235.

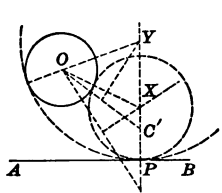


Construction. Let AB be the given line, O the centre of the given ⊙. Draw a straight line through O and P. At P erect a ⊥ to this line, meeting AB at C; then this ⊥ is a tangent to the given ⊙ (§ 253). Draw the bisectors of the ∠ ACP, BCP, and let them meet the line OP at the points D and E. With centre D and radius DP describe a ⊙; with centre E and radius EP describe a ⊙; these Ⓢ satisfy the given conditions. Q. E. F.

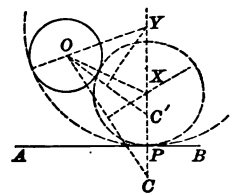
Discussion. If AB is tangent to the given ⊙, one of the two solutions is a ⊙ coinciding with the given ⊙; if AB is tangent at P, there is an indefinite number of solutions; if AB cuts the given ⊙, one solution will be a ⊙ touching the given ⊙ internally; if AB cuts the given ⊙, and also passes through P, the problem is impossible.

Ex. 237. *To construct a circle which shall touch a given line AB at P and also touch a given circle.*

Analysis. Suppose the problem solved, and let O be the centre of the given ⊙, X the centre of the required ⊙. One locus for X is the ⊥ to AB erected at P. Produce this ⊥, making PC equal to the radius of the given ⊙; then $OX = CX$. Therefore, another locus for X is the ⊥ bisector of the line joining O and C. The intersection of the two loci determines X. Q.E.F.



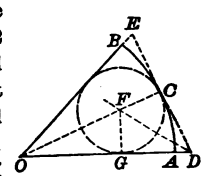
Discussion. Another solution is obtained by laying off PC' equal to the given radius upon the $\perp$ to AB and *on the same side* of AB as the given $\odot$, drawing OC', and erecting the $\perp$ bisector of OC'. The $\odot$ obtained by using C as the auxiliary point touches the given $\odot$ externally; while the $\odot$ obtained by using C' is touched by the given $\odot$ internally. If AB is supposed to move $\parallel$ to itself towards the given $\odot$, then the $\odot$ obtained by the use of C' becomes larger and larger; at the limit when AB touches the given $\odot$, the problem is impossible. If AB cuts the given circumference, the $\odot$ obtained by use of C' lies on the other side of AB, and touches the given $\odot$ externally. If AB cuts the given $\odot$, and P lies within the given $\odot$, then both the ⑤ touch the given $\odot$ internally.



Ex. 238. *To inscribe a circle in a given sector.*

Analysis. Let AOB be the given sector, and let the bisector of the $\angle AOB$ meet the arc AB at C. We know that the centre of the $\odot$ required must lie in OC. § 162

Draw a tangent to the given arc through C, and let it meet OA produced at D and OB produced at E; then we also know that the required $\odot$ will touch this tangent. § 221



Construction. Draw OC, the bisector of the $\angle AOB$. § 304

Through C draw a tangent meeting OA produced at D and OB produced at E. § 317

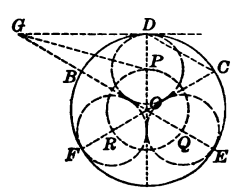
Bisect the $\angle ODE$, and let the bisector meet OC at F.

With centre F and radius FC describe a $\odot$.

This $\odot$ is the $\odot$ required. Q. E. F.

Ex. 239. *To construct within a given circle three equal circles, so that each shall touch the other two and also the given circle.*

Analysis. Suppose the problem solved, and let O be the centre of the given $\odot$, and P, Q, R the centres of the required ⑤. From the equality of the circles it is clear that their centres must be symmetrically arranged around O; and hence we see that the $\measuredangle$ at O formed by OP, OQ, OR are each 120°. We also see that these $\measuredangle$ are bisected by the radii OA, OB, OC, drawn

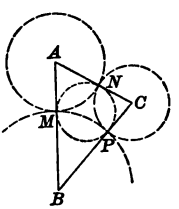


tangent to the three Ⓢ (§ 261); and therefore that OA, OB, OC also form at O ∡s each equal to 120°. The problem is now reduced to inscribing Ⓢ in sectors of 120°. Ex. 238

Construction. Draw a diameter AD. With centre D and radius DO equal to $\frac{1}{2}AD$, cut the circumference at B and C. Draw the diameters BOE, COF. Draw a tangent at D (§ 317), and produce OB to meet it at G. Bisect $\angle OGD$ (§ 304), and let the bisector meet OD at P. With centre O and radius OP, cut BE at Q, CF at R. With P, Q, R as centres, and radius PO, describe Ⓢ. These Ⓢ are the Ⓢ required. Q. E. F.

Ex. 240. *To describe circles about the vertices of a given triangle as centres, so that each shall touch the two others.*

Analysis. Suppose the problem solved, ABC the required △, and let the Ⓢ touch one another at M, N, P. Since $AM = AN$, and $CN = CP$ (§ 217), we infer that the three points M, N, P are the points of contact of the ⊙ inscribed in the △ ABC (§ 261), whence the construction is obvious.



Construction. Inscribe a ⊙ in the △ ABC (§ 315), and mark the points of contact M, N, P.

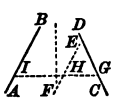
With centre A and radius AM describe a ⊙;
with centre B and radius BM describe a ⊙;
with centre C and radius CN describe a ⊙.
These Ⓢ are the Ⓢ required. Q. E. F.

Ex. 241. *To bisect the angle formed by two lines, without producing the lines to their point of intersection.*

Construction. Let AB, CD be the given lines.

Through any point E of CD draw $EF \parallel$ to BA. § 306

Upon the sides of the $\angle FEC$ take equal lengths EH, EG, and produce GH to meet AB at I.



Draw the ⊥ bisector of GI.

This line bisects the ∠ between the lines AB, CD.

Proof. The $\angle EGH = \angle EHG$. § 145

Also $\angle EHG = \angle BIG$. § 112

$\therefore \angle EGH = \angle BIG$. Ax. 1

$\therefore$ △ formed by the two given lines and the line GI is isosceles. § 147

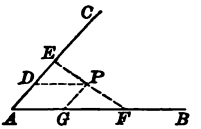
$\therefore$ the ⊥ bisector of GI bisects the ∠ between AB and CD. Ex. 35

Q. E. F.

Ex. 242. *To draw through a given point P between the sides of an angle BAC a line terminated by the sides of the angle and bisected at P.*

Analysis. Suppose the problem solved, and EF the required line. Draw $PD \parallel$ to BA, meeting AC at D. Then $AD = DE$. § 188

Hence, the point E may be determined.



Construction. Draw $PD \parallel$ to BA.

Upon AC take DE equal to AD.

Draw EP, and produce it to meet AB at F.

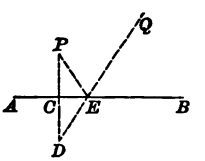
EF is the line required.

Proof. $PE = PF$. § 188

Q. E. F.

Ex. 243. *Given two points P, Q, and a line AB; to draw lines from P and Q which shall meet on AB and make equal angles with AB.*

Analysis. Let PE, QE be the lines required. Draw $PC \perp$ to AB, and let QE produced meet PC produced at D.



Since $\angle QEB = \angle DEC$, § 93

we have $\angle DEC = \angle PEC$. Ax. 1

$\therefore \triangle PCE = \triangle DCE$. § 142

$\therefore PC = CD$. § 128

Hence, the point E may be determined.

Construction. Draw $PC \perp$ to AB, and produce it to D, making CD equal to PC.

Draw QD, cutting AB at E, and draw PE.

PE and QE are the required lines.

Proof. The rt. $\triangle PCE$ and DCE are equal. § 144

For $CE = CE$, Iden.

and $PC = CD$. Const.

$\therefore \angle PEC = \angle DEC$. § 128

But $\angle DEC = \angle QEB$. § 93

$\therefore \angle PEC = \angle QEB$. Ax. 1

Q. E. F.

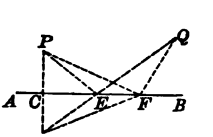
Discussion. If the line AB passes between P and Q, the solution is evidently the line joining P and Q.

Ex. 244. *To find the shortest path from P to Q which shall touch a line AB.*

Construction. Find the point in AB, such that PE and QE make equal $\measuredangle$s with AB. Ex. 243

Let F be any other point in AB.

Draw $PC \perp$ to AB, and produce it to D, making CD equal to PC.



Draw PF, QF, DE, DF.

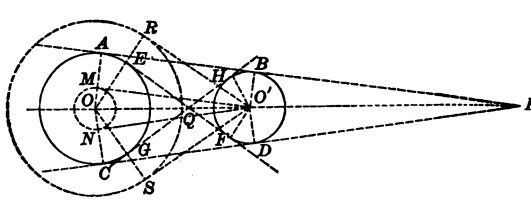
Then	$PE + EQ < PF + FQ.$	
Proof.	$\triangle PCE = \triangle DCE,$	
and	$\triangle PCF = \triangle DCF.$	§ 144
	$\therefore PE = DE,$	
and	$PF = DF.$	§ 128
	$\therefore PE + EQ = DE + EQ = DQ,$	
and	$PF + FQ = DF + FQ.$	Ax. 2
But	$DQ < DF + FQ.$	§ 49
	$\therefore PE + EQ < PF + FQ.$	

That is, the shortest path from P to Q which can be drawn so as to touch the line AB is the broken line whose two parts form equal $\measuredangle$ with AB. Q. E. F.

Ex. 245. *To draw a common tangent to two given circles.*

Analysis. Suppose the problem solved, and let O, O' be the centres of the Ⓢ, AB one of the common tangents; also let r and r' denote the radii of the Ⓢ. Draw OA, $O'B$; then OA and $O'B$ are $\perp$ to AB (§ 254), and, therefore, are $\parallel$ to each other. § 104

From A lay off upon AO a length AM equal to $O'B$, and draw MO'; then the figure $ABO'M$ is a $\square$ (§ 183), and a rectangle. And since $O'M$ is $\perp$ to OA, and $OM = r - r'$, therefore, $O'M$ will touch a ⊙ having O for centre and $r - r'$ for radius. This ⊙ can be described and $O'M$ drawn; after which the remainder of the construction is obvious.



Construction. With centre O and radius $r - r'$ describe a ⊙.

Draw to this ⊙ from O' the tangents $O'M$, $O'N'$. § 317

Draw OM, ON, and produce them to meet the given circumference at A and C.

Draw the radii $O'B \parallel$ to OA, and $O'D \parallel$ to OC.

Draw straight lines through A and B, and through C and D.

These lines are tangents to the given Ⓢ.

Proof.	MA is equal and $\parallel$ to $O'B.$	Const.
	$\therefore AB$ is $\parallel$ to $MO'.$	§ 183
Now	$\angle OMO' = 90°.$	§ 254

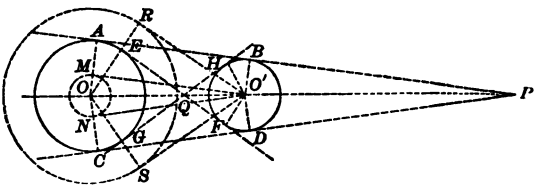
$\therefore \angle OAB = 90°$. § 107

$\therefore \angle O'BA = 90°$. § 115

$\therefore AB$ touches each ⊙. § 253

Similarly, CD may be shown to be a tangent. Q. E. F.

Discussion. Let d denote the distance OO' between the centres. If (as in the figure) $d > r + r'$, two more common tangents EF, GH may be drawn by the use of an auxiliary ⊙ having for radius $r + r'$ instead of



$r - r'$. The figure shows the details of their construction. AB and CD are called *external* tangents, EF and GH *internal* tangents. The external tangents meet at a point P which lies in the line of centres OO' produced. The internal tangents meet at a point Q which lies in the line of centres between O and O'. P is called the *direct centre of similitude*; Q is called *the inverse centre of similitude.*

If $d = r + r'$, the Ⓢ touch externally, and *three* common tangents can be drawn. Ex. 95

If $d < r + r'$ but $> r - r'$, the Ⓢ intersect, and *two* common tangents can be drawn. Ex. 95

If $d = r - r'$, the Ⓢ touch internally, and *one* common tangent can be drawn. Ex. 95

If $d < r - r'$, one ⊙ lies within the other without touching it, and the solution of the problem is impossible. Ex. 95

Ex. 246. *Find the fourth proportional to* 91, 65, *and* 133.

Let x = the fourth proportional required.

Then $91 : 65 = 133 : x.$ § 325

$$x = \frac{65 \times 133}{91} = 95.$$ § 327

Ex. 247. *Find the mean proportional between* 39 *and* 351.

Let x = the mean proportional required.

Then $39 : x = x : 351.$ § 326

$x^2 = 39 \times 351 = 3^2 \times 39^2.$ § 327

$x = 3 \times 39 = 117.$

Ex. 248. *Find the third proportional to* 54 *and* 3.

Let x = the third proportional required.

Then $54 : 3 = 3 : x.$ § 326

$$x = \frac{3 \times 3}{54} = \tfrac{1}{6}.$$ § 327

Ex. 249. *In a triangle* ABC, $AB = 12$, $AC = 14$, $BC = 13$. *Find the segments of* BC *made by the bisector of the angle* A.

Let M be the point in which the bisector of the $\angle A$ intersects BC.

Then $MC : MB = AC : AB = 14 : 12 = 7 : 6.$ § 348

$\therefore MC = \frac{7}{13}$ of $13 = 7$;

and $MB = \frac{6}{13}$ of $13 = 6.$

Ex. 250. *In a triangle* CAB, $CA = 6$, $CB = 12$, $AB = 15$. *Find the segments of* AB *made by the bisector of the angle* C.

Let M be the point in which the bisector of the $\angle C$ intersects AB.

Then $MA : MB = CA : CB = 6 : 12 = 1 : 2.$ § 348

$\therefore MA = \frac{1}{3}$ of $15 = 5$;

and $MB = \frac{2}{3}$ of $15 = 10.$

Ex. 251. *The base and altitude of a triangle are* 7 *feet* 6 *inches and* 5 *feet* 6 *inches, respectively. If the homologous base of a similar triangle is* 5 *feet* 6 *inches, find its homologous altitude.*

Let x = the length of the homologous altitude of the similar triangle.

Then 7 ft. 6 in. : 5 ft. 6 in. = 5 ft. 6 in. : x. § 361

$7\frac{1}{2} : 5\frac{1}{2} = 5\frac{1}{2} : x.$

$15 : 11 = 11 : 2x.$

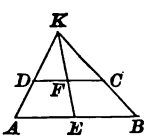
$30x = 121.$

$x = 4\frac{1}{30}.$ $\quad 4\frac{1}{30}$ ft. = 4 ft. $\frac{2}{5}$ in. *Ans.*

Ex. 252. *If the line joining the middle points of the bases of a trapezoid is produced, and the two legs are also produced, the three lines will meet in the same point.*

Let $ABCD$ be a trapezoid, having the sides AB and DC ∥. Let E be the middle point of AB and F the middle point of DC.

To prove that AD, EF, and BC, when produced, are concurrent.



Proof. $AE = EB$, and $DF = FC$. Hyp.

$\therefore AE : DF = EB : FC.$

The lines AD, EF, and BC are not parallel. § 165

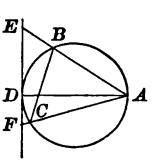
∴ the three lines are concurrent. § 363

Q. E. D.

Ex. 253. *AB and AC are chords drawn from any point A in the circumference of a circle, and AD is a diameter. The tangent to the circle at D intersects AB and AC at E and F, respectively. Show that the triangles ABC and AEF are similar.*

Let AB and AC be two chords drawn from the same point A of the circumference. Let AD be a diameter, and let EF, the tangent at D, intersect AB and AC produced at E and F, respectively.

To prove that the △ ABC and AEF are similar.



Proof. $\angle E$ is measured by $\frac{1}{2}$(arc ACD − arc BD). § 296

Since AD is a diameter, arc ACD = arc ABD.

$\therefore \angle E$ is measured by $\frac{1}{2}$(arc ABD − arc BD), that is, by $\frac{1}{2}$ arc AB.

But $\angle BCA$ is measured by $\frac{1}{2}$ arc AB. § 289

$\therefore \angle BCA = \angle E.$ Ax. 1

In the △ ABC and AEF,

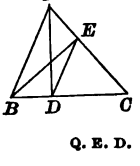
$\angle A$ is common, and $\angle BCA = \angle E$.

∴ the △ ABC and AEF are similar. § 355

Q. E. D.

Ex. 254. *AD and BE are two altitudes of the triangle CAB. Show that the triangles CED and CAB are similar.*

Let AD and BE be altitudes of the $\triangle CAB$.
To prove that the $\triangle CED$ and CAB are similar.



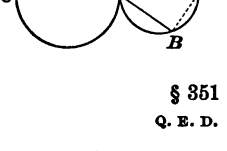
Proof. The rt. $\triangle CEB$ and ADC are similar. § 356
For the $\angle C$ is common.
$\therefore CE : CD = CB : CA.$ § 351
$\therefore$ the $\triangle CED$ and CAB are similar. § 357

Q. E. D.

Ex. 255. *If two circles are tangent to each other, the chords formed by a straight line drawn through the point of contact have the same ratio as the diameters of the circles.*

Let CE and ED be the diameters of the two circles which are tangent at E. Let AB be a line drawn through E and terminated by the circumferences.

To prove that $AE : EB = CE : ED$.

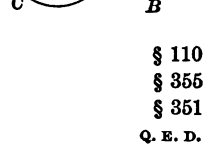


Proof. Draw AC and DB.
CE and ED form one straight line. § 265
AC is $\parallel$ to DB. Ex. 120
$\therefore \angle CAE = \angle EBD,$
and $\angle ACE = \angle EDB.$ § 110
$\therefore$ the $\triangle CAE$ and DBE are similar. § 355
$\therefore AE : EB = CE : ED.$ § 351

Q. E. D.

Ex. 256. *If two circles are tangent to each other externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circumferences are proportional.*

Let AB and CD be two lines drawn through E, the point of contact of the two circles which are tangent externally.



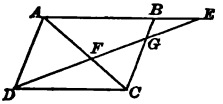
To prove that $AE : EB = CE : ED$.

Proof. Draw AC and DB.
AC is $\parallel$ to DB. Ex. 120
$\therefore \angle CAE = \angle EBD,$
and $\angle ACE = \angle EDB.$ § 110
$\therefore$ the $\triangle CAE$ and DBE are similar. § 355
$\therefore AE : EB = CE : ED.$ § 351

Q. E. D.

Ex. 257. *In a parallelogram $ABCD$ a line DE is drawn, meeting the diagonal AC in F, the side BC in G, and the side AB produced in E. Prove that $\overline{DF}^2 = FG \times FE$.*

Let $ABCD$ be a $\square$, and DE be drawn, meeting the diagonal AC in F, the side BC in G, and the side AB produced in E.



To prove that $\overline{DF}^2 = FG \times FE$.

Proof. The $\triangle AEF$ and DFC are similar. § 355

For $\angle AEF = \angle FDC$,

and $\angle FAE = \angle FCD$. § 110

$\therefore DF : FE = FC : AF$. § 351

The $\triangle AFD$ and GFC are similar. § 355

For $\angle AFD = \angle GFC$, § 93

and $\angle DAF = \angle FCG$. § 110

$\therefore FG : DF = FC : AF$. § 351

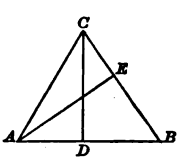
$\therefore DF : FE = FG : DF$. Ax. 1

$\therefore \overline{DF}^2 = FG \times FE$. § 327

Q. E. D.

Ex. 258. *Two altitudes of a triangle are inversely proportional to the corresponding bases.*

In the $\triangle ABC$, let AE and CD be the altitudes drawn to BC and AB, respectively.



To prove that $AE : CD = AB : BC$.

Proof. The rt. $\triangle AEB$ and CDB have the common acute $\angle B$.

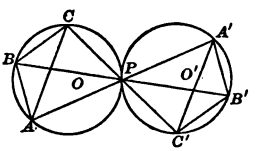
$\therefore \triangle AEB$ and CDB are similar. § 356

$\therefore AE : CD = AB : BC$. § 351

Q. E. D.

Ex. 259. *Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C', respectively. Prove that the triangles ABC, $A'B'C'$ are similar.*

In the ⊙ whose centres are O and O', let AA', BB', and CC' be drawn through P, the point of contact, and let AC, BC, AB, $A'C'$, $B'C'$, and $A'B'$ be drawn.



To prove that the $\triangle ABC$ and $A'B'C'$ are similar.

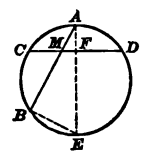
Proof. AC, BC, AB are ∥ to $C'A'$, $C'B'$, $B'A'$, respectively. Ex. 120

$\therefore \triangle ABC$ and $A'B'C'$ are similar. § 359

Q. E. D.

Ex. 260. *Two chords AB, CD intersect at M, and A is the middle point of the arc CD. Prove that the product $AB \times AM$ is constant if the chord AB is made to turn about the fixed point A.*

Let AB and CD be two chords intersecting at M, and let A be the middle point of the arc CD.



To prove that $AB \times AM$ is constant.

Proof. Draw the diameter AE and draw BE.

AF is $\perp$ to CD. § 245

$\triangle ABE$ is a rt. $\triangle$. § 290

The rt. $\triangle AFM$ and ABE have the common $\angle MAF$.

$\therefore \triangle AFM$ and ABE are similar. § 356

$\therefore AF : AB = AM : AE$. § 351

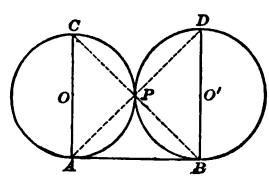
$\therefore AF \times AE = AB \times AM$. § 327

But $AF \times AE$ is constant.

$\therefore AB \times AM$ is constant. Q. E. D.

Ex. 261. *If two circles touch each other, their common external tangent is the mean proportional between their diameters.*

Let O and O' be the centres of two ⊙ that touch at P, and AC and BD their diameters. Let AB be their common external tangent.



To prove that $AC : AB = AB : BD$.

Proof. Draw AP, CP, BP, DP.

AC and BD are $\perp$ to AB. § 254

Now $\angle APB$ = a rt. $\angle$. Ex. 100

Also $\angle APC$ = a rt. $\angle$. § 290

$\therefore CPB$ is a straight line. § 90

In like manner, DPA is a straight line.

Now the rt. $\triangle ACB$ and BAD are similar. § 356

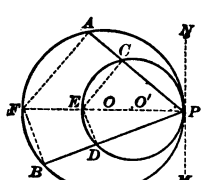
For $\angle C = \angle DAB$. §§ 289, 295

$\therefore AC : AB = AB : BD$. § 351

Q. E. D.

Ex. 262. *If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.*

Let the ⊙ whose centres are O and O' be tangent internally at P, and chords PA and PB be drawn, cutting the circumference of the smaller ⊙ at C and D.



To prove that $PC : PA = PD : PB$.

Proof. Draw the tangent MN at the point P.

Draw $PF \perp$ to MN.

Then PF passes through O and O'. § 255

Draw EC, ED, FA, and FB.

$\measuredangle PCE$ and PAF are rt. $\measuredangle$. § 290

The rt. $\triangle PCE$ and PAF have the $\angle EPC$ common.

$\therefore \triangle PCE$ and PAF are similar. § 356

$\therefore PC : PA = PE : PF$. § 351

Similarly, $PD : PB = PE : PF$.

$\therefore PC : PA = PD : PB$. Ax. 1

Q. E. D.

Ex. 263. *In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.*

Let $ABCD$ be an inscribed quadrilateral, and AC and DB its diagonals.

To prove that $AC \times DB = DC \times AB + AD \times BC$.

Proof. Draw DE, making $\angle CDE$ equal to $\angle ADB$.

Add to each the $\angle BDE$.

Then the $\triangle ADE$ and BDC are similar. § 355

For $\angle ADE = \angle BDC$, Ax. 2

and $\angle DAC = \angle DBC$. § 293

$\therefore AD : DB = AE : BC$. § 351

$\therefore AD \times BC = DB \times AE$. § 327

The $\triangle DEC$ and DAB are similar. § 355

For $\angle CDE = \angle ADB$, Const.

and $\angle DCE = \angle ABD$. § 293

$\therefore DC : DB = EC : AB$. § 351

$\therefore DC \times AB = DB \times EC$. § 327

Add the two resulting equations.

$\therefore DC \times AB + AD \times BC = DB(AE + EC) = DB \times AC$. Ax. 2

Q. E. D.

Ex. 264. *Two isosceles triangles with equal vertical angles are similar.*

In the $\triangle ABC$ and $A'B'C'$, let AB be equal to AC, $A'B'$ to $A'C'$, and $\angle A$ to $\angle A'$.

To prove that the $\triangle ABC$ and $A'B'C'$ are similar.

Proof. $AB = AC$,

and $A'B' = A'C'$. Hyp.

$\therefore AB : A'B' = AC : A'C'$.

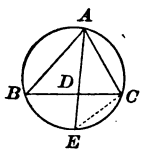
Also $\angle A = \angle A'$. Hyp.

$\therefore \triangle ABC$ and $A'B'C'$ are similar. § 357

Q. E. D.

Ex. 265. *The bisector of the vertical angle A of the triangle ABC intersects the base at D and the circumference of the circumscribed circle at E. Show that* $AB \times AC = AD \times AE$.

Let $\triangle ABC$ be inscribed in a circle, and let AE, the bisector of the $\angle A$, intersect BC at D and the circumference at E.



To prove that $AB \times AC = AD \times AE$.

Proof. Draw CE.

The $\triangle BAD$ and EAC are similar. § 355

For $\angle BAD = \angle EAC$, Hyp.

and $\angle ABD = \angle AEC$. § 293

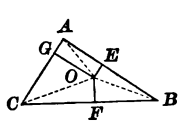
$\therefore AB : AE = AD : AC$. § 351

$\therefore AB \times AC = AD \times AE$. § 327

Q. E. D.

Ex. 266. *If OE, OF, OG are the perpendiculars from any point O within the triangle ABC upon the sides AB, BC, CA, respectively, show that* $\overline{AE}^2 + \overline{BF}^2 + \overline{CG}^2 = \overline{EB}^2 + \overline{FC}^2 + \overline{GA}^2$.

Let OE, OF, OG be the perpendiculars from any point O within the $\triangle ABC$ upon the sides AB, BC, CA, respectively.



To prove that

$\overline{AE}^2 + \overline{BF}^2 + \overline{CG}^2 = \overline{EB}^2 + \overline{FC}^2 + \overline{GA}^2$.

Proof. Draw OA, OB, OC.

Now $\overline{OA}^2 = \overline{AE}^2 + \overline{OE}^2$,

and $\overline{OA}^2 = \overline{OG}^2 + \overline{GA}^2$. § 371

$\therefore \overline{AE}^2 + \overline{OE}^2 = \overline{OG}^2 + \overline{GA}^2$. Ax. 1

In like manner, $\overline{BF}^2 + \overline{OF}^2 = \overline{EB}^2 + \overline{OE}^2$,

and $\overline{CG}^2 + \overline{OG}^2 = \overline{FC}^2 + \overline{OF}^2$.

Add these three equations.

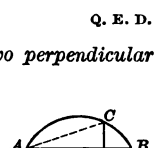
Then $\overline{AE}^2 + \overline{OE}^2 + \overline{BF}^2 + \overline{OF}^2 + \overline{CG}^2 + \overline{OG}^2$

$= \overline{OG}^2 + \overline{GA}^2 + \overline{EB}^2 + \overline{OE}^2 + \overline{FC}^2 + \overline{OF}^2$. Ax. 2

$\therefore \overline{AE}^2 + \overline{BF}^2 + \overline{CG}^2 = \overline{EB}^2 + \overline{FC}^2 + \overline{GA}^2$. Q. E. D.

Ex. 267. *The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.*

Let the $\perp$ chords AB and CD intersect at P. Draw the diameter BE.

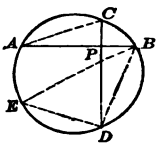


To prove that $\overline{PA}^2 + \overline{PC}^2 + \overline{PB}^2 + \overline{PD}^2 = \overline{BE}^2$.

Proof. Draw AC, ED, and BD.

Then arc AC + arc BD = arc BDE. Ex. 98

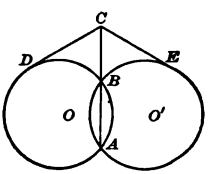
Subtract arc BD from both sides.



Then	arc AC = arc ED.	Ax. 3
	$\therefore$ chord AC = chord ED.	§ 241
Now	$\overline{ED}^2 + \overline{BD}^2 = \overline{BE}^2$.	§ 371
	$\therefore \overline{AC}^2 + \overline{BD}^2 = \overline{BE}^2$.	
But	$\overline{AC}^2 = \overline{PA}^2 + \overline{PC}^2$,	
and	$\overline{BD}^2 = \overline{PB}^2 + \overline{PD}^2$.	§ 371
Substituting,	$\overline{PA}^2 + \overline{PC}^2 + \overline{PB}^2 + \overline{PD}^2 = \overline{BE}^2$.	Q. E. D.

Ex. 268. *The tangents to two intersecting circles drawn from any point in their common chord produced are equal.*

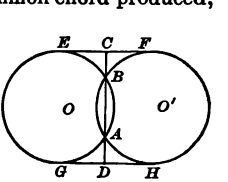
Let O and O' be two intersecting ⊙, AB be their common chord, and CD and CE be tangents to O and O' from any point C of AB produced.



To prove that	$CD = CE$.	
Proof.	$\overline{CD}^2 = CA \times CB$,	
and	$\overline{CE}^2 = CA \times CB$.	§ 381
	$\therefore \overline{CD}^2 = \overline{CE}^2$.	Ax. 1
	$\therefore CD = CE$.	Q. E. D.

Ex. 269. *The common chord of two intersecting circles, if produced, will bisect their common tangents.*

Let O and O' be two intersecting ⊙, DC their common chord produced, and EF and GH their common tangents.

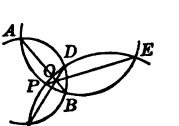


To prove that DC bisects EF and GH.

Proof.	$\overline{EC}^2 = CB \times CA$,	
and	$\overline{FC}^2 = CB \times CA$.	§ 381
	$\therefore \overline{EC}^2 = \overline{FC}^2$.	Ax. 1
	$\therefore EC = FC$.	
In like manner,	$DG = DH$.	Q. E. D.

Ex. 270. *If three circles intersect one another, the common chords all pass through the same point.*

Let the ⊙ ABC, ABE, and CDE intersect one another, and AB and CD meet at O. Draw EO. Suppose that EO produced meets ABE at P and CDE at Q.



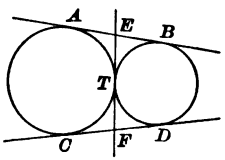
To prove that P and Q coincide.

Proof. In ⊙ ABC,	$OA \times OB = OC \times OD$.	§ 378
In ⊙ ABE,	$OA \times OB = OE \times OP$.	§ 378
In ⊙ CDE,	$OC \times OD = OE \times OQ$.	§ 378
	$\therefore OE \times OP = OE \times OQ$.	Ax. 1
	$\therefore OP = OQ$.	

Therefore, EP is the common chord of ⊙ ABE and CDE, and passes through the point O. Q. E. D.

Ex. 271. *If two circles are tangent to each other, the common internal tangent bisects the two common external tangents.*

Let EF, the common internal tangent to the two tangent circles, intersect AB and CF, the common external tangents, in the points E and F.



To prove that $EA = EB$, and $FC = FD$.

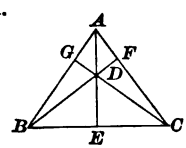
Proof.

	$EA = ET$,	
and	$EB = ET$.	§ 261
	$\therefore EA = EB$.	Ax. 1
In like manner,	$FC = FD$.	Q. E. D.

Ex. 272. *If the perpendiculars from the vertices of the triangle ABC upon the opposite sides intersect at D, show that*

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Let AE, BF, CG be the ⊥s from the vertices of the $\triangle ABC$ upon the opposite sides, and let D be their point of intersection.



To prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2$.

Proof. Now $\overline{AB}^2 = \overline{AE}^2 + \overline{BE}^2$,

and $\overline{AC}^2 = \overline{AE}^2 + \overline{CE}^2$. § 371

$\therefore \overline{AB}^2 - \overline{AC}^2 = \overline{BE}^2 - \overline{CE}^2$. Ax. 3

Again, $\overline{BD}^2 = \overline{DE}^2 + \overline{BE}^2$,

and $\overline{CD}^2 = \overline{DE}^2 + \overline{CE}^2$. § 371

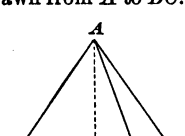
$\therefore \overline{BD}^2 - \overline{CD}^2 = \overline{BE}^2 - \overline{CE}^2$. Ax. 3

$\therefore \overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2$. Ax. 1

Q. E. D.

Ex. 273. *In an isosceles triangle, the square of a leg is equal to the square of any line drawn from the vertex to the base, increased by the product of the segments of the base.*

In the $\triangle ABC$, let AB equal AC, AE be any line drawn from A to BC.



To prove that $\overline{AC}^2 = \overline{AE}^2 + BE \times EC$.

Proof. Draw the altitude AD.

Since AE is any line drawn to BC, in general, the ∡s AEB and AEC are supplementary unequal ∡s.

Suppose the $\angle AEC$ is obtuse.

Then $\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 + 2\,EC \times DE$. § 376

That is, $\overline{AC}^2 = \overline{AE}^2 + EC\,(EC + 2\,DE)$.

Now $BD = DC$. § 149

That is, $BD = EC + DE$.

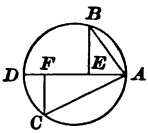
$\therefore BD + DE = EC + 2\,DE$. Ax. 2

That is, $BE = EC + 2\,DE$.

$\therefore \overline{AC}^2 = \overline{AE}^2 + BE \times EC$. Q. E. D.

Ex. 274. *The squares of two chords drawn from the same point in a circumference have the same ratio as the projections of the chords on the diameter drawn from the same point.*

Let AB and AC be two chords of the $\odot ABC$ drawn from the same point A in the circumference. Let AE and AF be the projections of AB and AC on the diameter drawn from A.



To prove that $\overline{AB}^2 : \overline{AC}^2 = AE : AF$.

Proof. $\overline{AB}^2 = AD \times AE$,

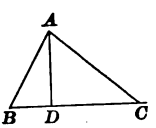
and $\overline{AC}^2 = AD \times AF$. § 370

$\therefore \overline{AB}^2 : \overline{AC}^2 = AD \times AE : AD \times AF$.

$\therefore \overline{AB}^2 : \overline{AC}^2 = AE : AF$. Q. E. D.

Ex. 275. *The difference of the squares of two sides of a triangle is equal to the difference of the squares of the segments of the third side, made by the perpendicular on the third side from the opposite vertex.*

Let AD be the perpendicular from A to the base BC of the $\triangle ABC$. Let AC be greater than AB.



To prove that $\overline{AC}^2 - \overline{AB}^2 = \overline{DC}^2 - \overline{BD}^2$.

Proof. $\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2$,

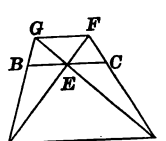
and $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2$. § 371

$\therefore \overline{AC}^2 - \overline{AB}^2 = \overline{DC}^2 - \overline{BD}^2$. Ax. 3

Q. E. D.

Ex. 276. *E is the middle point of BC, one of the parallel sides of the trapezoid $ABCD$; AE and DE produced meet DC and AB produced at F and G, respectively. Show that FG is parallel to DA.*

Let E be the middle point of BC, one of the $\parallel$ sides of the trapezoid $ABCD$. Let AE and DE produced meet DC and AB at F and G, respectively.



To prove that FG is $\parallel$ to DA.

Proof. The $\triangle AGD$ and BGE are similar. § 355

For $\angle GAD = \angle GBE$,

and $\angle GDA = \angle GEB$. § 112

$\therefore DG : EG = AD : BE$. § 351

The $\triangle AFD$ and EFC are similar. § 355

For $\angle FAD = \angle FEC$, and $\angle FDA = \angle FCE$. § 112

$\therefore AF : EF = AD : CE$. § 351

Now $BE = CE$. Hyp.

$\therefore AD : BE = AD : CE$.

$\therefore DG : EG = AF : EF$. Ax. 1

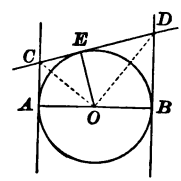
$\therefore DG - EG : EG = AF - EF : EF$. § 333

That is, $ED : EG = EA : EF$.

The $\triangle AED$ and FEG are similar. § 357
For $\angle AED = \angle FEG$. § 95
$\therefore \angle EGF = \angle EDA$. § 351
$\therefore FG$ is $\parallel$ to DA. § 111
Q. E. D.

Ex. 277. *If two tangents are drawn to a circle at the extremities of a diameter, the portion of a third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.*

Let AC and BD be tangent to the circle whose centre is O at the extremities of a diameter AB. Let CD be a third tangent to the circle at the point E, and OE be a radius of the circle.

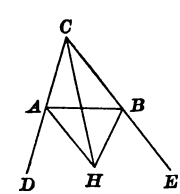


To prove that $EC \times ED = \overline{OE}^2$.

Proof. AC and BD are $\perp$ to AB. § 254
$\therefore AC$ and BD are $\parallel$. § 104
$\therefore \angle ACE + \angle BDE = 2$ rt. $\angle$s. § 115
Draw OC and OD.
Then $\angle ACO = \angle ECO$,
and $\angle BDO = \angle EDO$. § 261
$\therefore \angle ECO + \angle EDO = 1$ rt. $\angle$. Ax. 7
$\therefore \angle COD$ is a rt. $\angle$. § 131
Now OE is $\perp$ to CD. § 254
$\therefore CE : OE = OE : ED$. § 367
$\therefore CE \times ED = \overline{OE}^2$. § 327
Q. E. D.

Ex. 278. *If two exterior angles of a triangle are bisected, the line drawn from the point of intersection of the bisectors to the opposite angle of the triangle bisects that angle.*

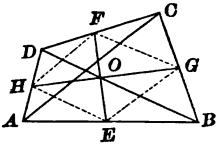
Let AH bisect the exterior $\angle BAD$ of the $\triangle ABC$, and BH bisect the exterior $\angle ABE$.



To prove that HC bisects the $\angle ACB$.

Proof. Since H lies in the bisector of the $\angle BAD$, H is equidistant from CA and AB. § 162
Since H lies in the bisector of the $\angle ABE$, H is equidistant from AB and CB. § 162
$\therefore H$ is equidistant from CA, AB, and CB.
Now the locus of points equidistant from CA and CB is the bisector of the $\angle ACB$. § 162
$\therefore HC$, having the two points C and H in common with this bisector, coincides with this bisector. § 47
That is, HC bisects the $\angle ACB$. Q. E. D.

Ex. 279. *The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines that join the middle points of the opposite sides.*



Let $ABCD$ be a quadrilateral, AC and BD its diagonals, and let EF and GH intersecting at O join the middle points of the opposite sides.

To prove that $\overline{AC}^2 + \overline{BD}^2 = 2(\overline{EF}^2 + \overline{GH}^2)$.

Proof. Draw EG, GF, FH, HE.

Now $AC = 2\,EG$. § 189

$\therefore \overline{AC}^2 = 4\,\overline{EG}^2$.

In like manner, $\overline{BD}^2 = 4\,\overline{GF}^2$.

$\therefore \overline{AC}^2 + \overline{BD}^2 = 4(\overline{EG}^2 + \overline{GF}^2)$. Ax. 2

But $EGFH$ is a $\square$. Ex. 74

$\therefore OG = OH$, and $OE = OF$. § 184

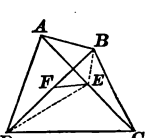
Now $\overline{EG}^2 + \overline{GF}^2 = 2\,\overline{OE}^2 + 2\,\overline{OG}^2$. § 377

$\therefore \overline{AC}^2 + \overline{BD}^2 = 4(2\,\overline{OE}^2 + 2\,\overline{OG}^2)$

$= 2(4\,\overline{OE}^2 + 4\,\overline{OG}^2)$.

$\therefore \overline{AC}^2 + \overline{BD}^2 = 2(\overline{EF}^2 + \overline{GH}^2)$. Q. E. D.

Ex. 280. *The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the middle points of the diagonals.*



Let $ABCD$ be a quadrilateral, and EF the line joining the middle points of the diagonals AC and BD.

To prove that $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$.

Proof. Draw BE and DE.

$$\overline{AB}^2 + \overline{BC}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{BE}^2, \quad \text{§ 377}$$

and $$\overline{CD}^2 + \overline{DA}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{DE}^2. \quad \text{§ 377}$$

Adding these equalities,

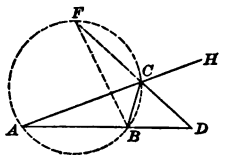
$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 2(\overline{BE}^2 + \overline{DE}^2). \quad \text{Ax. 2}$$

But $$\overline{BE}^2 + \overline{DE}^2 = 2\left(\frac{BD}{2}\right)^2 + 2\,\overline{EF}^2. \quad \text{§ 377}$$

$$\therefore \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 4\left(\frac{BD}{2}\right)^2 + 4\,\overline{EF}^2$$

$$= \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2. \quad \text{Q. E. D.}$$

Ex. 281. *The square of the bisector of an exterior angle of a triangle is equal to the product of the external segments determined by the bisector upon one of the sides diminished by the product of the other two sides.*



Let ABC be a $\triangle$, and CD be the bisector of the exterior $\angle BCH$, meeting AB produced at D.

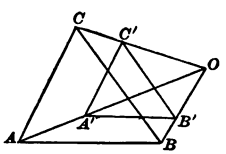
To prove that $\overline{CD}^2 = AD \times BD - AC \times BC$.

Proof. Circumscribe a $\odot$ about the $\triangle ABC$, produce DC to meet the circumference in F, and draw BF.

In $\triangle ACD$ and FCB,

	$\angle BAC = \angle BFC$,	§ 293
	$\angle FCA = \angle HCD$.	§ 93
But	$\angle HCD = \angle BCD$.	Hyp.
	$\therefore \angle FCA = \angle BCD$.	Ax. 1
	Add to each the $\angle ACB$.	
Then	$\angle FCB = \angle ACD$.	Ax. 2
	$\therefore \triangle ACD$ and FCB are similar.	§ 355
	$\therefore CD : BC = AC : FC$.	§ 351
	$\therefore FC \times CD = AC \times BC$.	§ 327
Now	$AD \times BD = FD \times CD$	§ 382
	$= (FC + CD)\,CD$	
	$= FC \times CD + \overline{CD}^2$.	
But	$FC \times CD = AC \times BC$.	
	$\therefore AD \times BD = AC \times BC + \overline{CD}^2$.	
Transposing,	$\overline{CD}^2 = AD \times BD - AC \times BC$.	Q. E. D.

Ex. 282. *If a point O is joined to the vertices of a triangle ABC; through any point A' in OA a line parallel to AB is drawn, meeting OB at B'; through B' a line parallel to BC, meeting OC at C'; and C' is joined to A'; the triangle $A'B'C'$ is similar to the triangle ABC.*



Let ABC be a $\triangle$, O any point; let OA, OB and OC be drawn; from A', any point in OA, let $A'B'$ be drawn $\parallel$ to AB; let $B'C'$ be drawn $\parallel$ to BC; join C' and A'.

To prove that the $\triangle ABC$ and $A'B'C'$ are similar.

Proof. In the $\triangle ABC$ and $A'B'C'$,

$\angle ABC = \angle A'B'C'$. § 176

In the $\triangle AOB$ and $A'O'B'$,

$\angle OB'A' = \angle OBA$, and $\angle OA'B' = \angle OAB$. § 112

$\therefore$ $\triangle AOB$ and $A'OB'$ are similar. § 355

$\therefore OB' : OB = A'B' : AB.$ § 351

In the same way, $\triangle OB'C'$ and OBC may be proved similar.

$\therefore OB' : OB = B'C' : BC.$ § 351

$\therefore A'B' : AB = B'C' : BC.$ Ax. 1

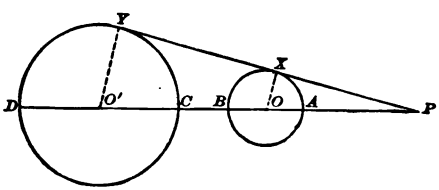
$\therefore \triangle ABC$ and $A'B'C'$ are similar. § 357

Q. E. D.

Ex. 283. *If the line of centres of two circles meets the circumferences at the consecutive points A, B, C, D, and meets the common external tangent at P, then* $PA \times PD = PB \times PC.$

In the ⊙ whose centres are O and O', let the line of centres meet the common external tangent PXY at P and cut the circumferences at A, B, C, D.

To prove that $PA \times PD = PB \times PC.$



Proof. Draw the radii OX and $O'Y$.

$\triangle PXO$ and PYO' are rt. $\triangle$. § 254

The rt. $\triangle PXO$ and PYO' are similar. § 356

For the $\angle YPO'$ is common.

$\therefore PO : OX = PO' : O'Y.$ § 351

By composition and division,

$PO + OX : PO - OX = PO' + O'Y : PO' - O'Y.$ § 334

But $PO + OX = PB$, $PO - OX = PA$, $PO' + O'Y = PD$,

and $PO' - O'Y = PC.$

$\therefore PB : PA = PD : PC.$

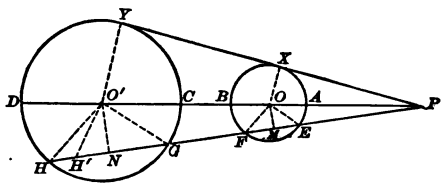
$\therefore PA \times PD = PB \times PC.$ § 327

Q. E. D.

Ex. 284. *The line of centres of two circles meets the common external tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that* $PE \times PH = PF \times PG.$

In the ⊙ whose centres are O and O', let the line of centres meet the common external tangent PXY at P, and cut the circumferences at A, B, C, D; let the secant PH be drawn, cutting the circumferences at E, F, G, H.

To prove that $PE \times PH = PF \times PG$.



Proof. Draw the radii OX, OE, OF, $O'Y$, $O'G$, and $O'H$; and draw OM and $O'N \perp$ to PH.

The rt. △ POM and $PO'N$ are similar. § 356

For the $\angle O'PN$ is common.

$\therefore PO : PO' = OM : O'N$. § 351

But $PO : PO' = OX : O'Y = OF : O'H$. Ex. 283

$\therefore OM : O'N = OF : O'H$. Ax. 1

Draw $O'H' \parallel$ to OF.

Then rt. △ OMF and $O'NH'$ are similar. § 356

For $\angle OFM = \angle O'H'N$. § 176

$\therefore OM : O'N = OF : O'H'$. § 351

$\therefore O'H = O'H'$ and coincides with it. $\therefore O'H$ is $\parallel$ to OF.

$\therefore PO : PO' = PF : PH$. § 342

In the same way it may be proved that

$PO : PO' = PE : PG$.

$\therefore PE : PG = PF : PH$. Ax. 1

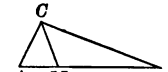
$\therefore PE \times PH = PF \times PG$. § 327

Q. E. D.

Ex. 285. *If a line drawn from a vertex of a triangle divides the opposite side into segments proportional to the adjacent sides, the line bisects the angle at the vertex.*

Let CM be a line from the vertex of the $\triangle ABC$ to the base AB, drawn so that $AM : MB = AC : CB$.

To prove that CM bisects $\angle ACB$.



Proof. The bisector of the $\angle ACB$ must pass through M. § 348

Therefore, CM, having two points in common with the bisector of the $\angle ACB$, must coincide with that bisector. § 47

$\therefore CM$ bisects the $\angle ACB$. Q. E. D.

Ex. 286. *Divide a line* 12 *inches long into three parts proportional to the numbers* 3, 5, 7.

$3 + 5 + 7 = 15.$

$\frac{3}{15}$ of 12 inches $= \frac{1}{5}$ of 12 inches $= 2\frac{2}{5}$ inches.

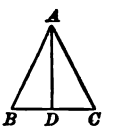
$\frac{5}{15}$ of 12 inches $= \frac{1}{3}$ of 12 inches $= 4$ inches.

$\frac{7}{15}$ of 12 inches $= 5\frac{3}{5}$ inches.

$2\frac{2}{5}$ inches, 4 inches, $5\frac{3}{5}$ inches. *Ans.*

Ex. 287. *The square of the altitude of an equilateral triangle is equal to three fourths of the square of one side of the triangle.*

Let AD be the altitude of the equilateral $\triangle ABC$.



To prove that $\overline{AD}^2 = \frac{3}{4}\overline{AC}^2$.

Proof.

$\overline{AD}^2 = \overline{AC}^2 - \overline{DC}^2$. § 372

But $DC = \frac{1}{2}BC = \frac{1}{2}AC$. § 149

$\therefore \overline{DC}^2 = \frac{1}{4}\overline{AC}^2$.

$\therefore \overline{AD}^2 = \overline{AC}^2 - \frac{1}{4}\overline{AC}^2$.

$\therefore \overline{AD}^2 = \frac{3}{4}\overline{AC}^2$.

Q. E. D.

Ex. 288. *Construct* x, *if* (1) $x = \frac{ab}{c}$, (2) $x = \frac{a^2}{c}$.

Special Cases: (1) $a = 2$, $b = 3$, $c = 4$; (2) $a = 3$, $b = 7$, $c = 11$; (3) $a = 2$, $c = 3$; (4) $a = 3$, $c = 5$; (5) $a = 2c$.

(1) $x = \frac{ab}{c}$.

$cx = ab$.

$\therefore c : a = b : x$. § 329

Hence, find x, the fourth proportional to c, a, and b. § 386

(2) $x = \frac{a^2}{c}$.

$cx = a^2$.

$\therefore c : a = a : x$. § 329

Hence, find x, the third proportional to c and a. § 387

Special Cases: (1) $a = 2$, $b = 3$, $c = 4$.

$4 : 2 = 3 : x$.

$\therefore x = 1\frac{1}{2}$.

(2) $a = 3$, $b = 7$, $c = 11$.

$11 : 3 = 7 : x$.

$\therefore x = 1\frac{10}{11}$.

(3) $a = 2$, $c = 3$.

$3 : 2 = 2 : x$.

$\therefore x = \frac{4}{3}$.

(4) $a = 3,\ c = 5.$

$5 : 3 = 3 : x.$

$\therefore x = 1\frac{4}{5}.$

(5) $a = 2\,c.$

$c : 2\,c = 2\,c : x.$

$\therefore x = 4\,c.$ Q. E. F.

Ex. 289. *Construct x, if $x = \sqrt{ab}$.*

Special Cases: (1) $a = 2,\ b = 3$; (2) $a = 1,\ b = 5$; (3) $a = 3,\ b = 7$.

Find x, the mean proportional between a and b. § 388

Special Cases: (1) $a = 2,\ b = 3$.

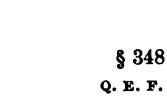
Using a convenient unit, find x, the mean proportional between 2 and 3. § 388

$x = \sqrt{6}.$

(2) Like (1). $x = \sqrt{5}.$

(3) Like (1). $x = \sqrt{21}.$ Q. E. F.

Ex. 290. *To divide one side of a given triangle into segments proportional to the adjacent sides.*



Let ABC be the given $\triangle$.

To divide AB into segments proportional to AC and BC.

Construction. Draw CM bisecting the $\angle ACB$.

Then M is the division point required.

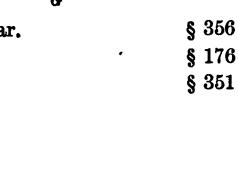
Proof. $AM : MB = AC : BC.$ § 348

Q. E. F.

Ex. 291. *To find in one side of a given triangle a point whose distances from the other sides shall be to each other in the given ratio $m : n$.*

Let ABC be a $\triangle$.

To find a point in AB whose distances from AC and BC shall be in the given ratio $m : n$.



Construction. Draw AG equal to m $\perp$ to AC, GH equal to n $\perp$ to BC, and $HO \parallel$ to BC.

Draw OG. From C draw $CD \parallel$ to OG; draw $DE \perp$ to AC, and $DF \perp$ to BC.

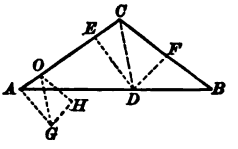
D is the point required.

Proof. The rt. $\triangle GHO$ and DFC are similar. § 356

For $\angle GOH = \angle DCF.$ § 176

$\therefore GH : DF = OG : CD.$ § 351

In like manner, the $\triangle AGO$ and EDC are similar.



$\therefore AG : DE = OG : CD.$ § 351

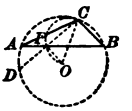
$\therefore AG : DE = GH : DF.$ Ax. 1

By alternation, $AG : GH = DE : DF.$ § 330

But $AG = m$, and $GH = n.$ Const.

$\therefore DE : DF = m : n.$ Q. E. F.

Ex. 292. *Given an obtuse triangle; to draw a line from the vertex of the obtuse angle to the opposite side which shall be the mean proportional between the segments of that side.*



Let ABC be a $\triangle$ with obtuse $\angle$ at C.

To draw a line from C to AB which shall be the mean proportional between the segments of AB.

Construction. Circumscribe a $\odot$ about ABC, and draw the radius OC; upon OC as a diameter describe a semicircle intersecting AB at P.

Draw CP, and prolong it to meet the circumference at D. Draw OP.

CP is the line required.

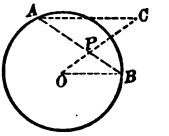
Proof. $\angle OPC$ is a rt. $\angle$. § 290

$\therefore CP = PD.$ § 245

Now $AP : CP = PD : PB.$ § 378

$\therefore AP : CP = CP : PB.$ Q. E. F.

Ex. 293. *Through a given point P within a given circle to draw a chord AB so that $AP : BP$ shall equal the given ratio $m : n$.*



Let P be the given point.

To draw a chord AB through P so that $AP : BP$ shall equal $m : n$.

Construction. Draw OP and prolong it to C, so that

$OP : PC = n : m.$

Find x, the fourth proportional to n, m, and the radius of the $\odot$. § 386

With C as centre and x as radius, cut the circumference at A.

Draw CA, and draw $BO \parallel$ to CA. Draw AP and BP.

APB is a straight line, and the chord required.

Proof. $OP : PC = n : m = OB : CA.$ Const.

$\therefore OP : OB = PC : CA.$ § 330

$\angle BOP = \angle PCA.$ § 110

$\therefore \triangle BOP$ and ACP are similar. § 357

$\therefore \angle BPO = \angle CPA.$ § 351

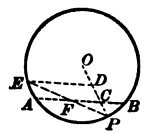
Since OPC is a straight line, APB is a straight line.

Also $AP : BP = PC : OP = m : n.$ Q. E. F.

Ex. 294. *To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.*

Let P be a point in the arc AB.

To draw a chord through P which shall be bisected by the chord AB.



Construction. Draw the radius OP intersecting AB at C; take CD equal to CP.

Draw $DE \parallel$ to BA, and draw EP.

EP is the chord required.

Proof.

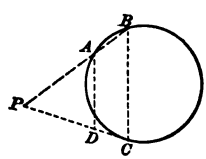
$CP = CD.$	Const.
$\therefore PF = FE.$	§ 188

Q. E. F.

Ex. 295. *To draw through a given external point P a secant PAB to a given circle so that the ratio PA : AB shall equal the given ratio m : n.*

Let P be a point exterior to the $\odot ABC$.

To draw through P a secant PAB so that $PA : AB = m : n$.



Construction. Draw the tangent PC.

Divide PC so that $PD : DC = m : n$. § 385

Find x, the mean proportional between PD and PC. § 388

With P as a centre with a radius equal to x, cut the circumference at A. Draw PAB.

PAB is the secant required.

Proof. Draw AD and BC.

	$PD : PA = PA : PC.$	Const.
But	$PA : PC = PC : PB.$	§ 381
	$\therefore PD : PA = PC : PB.$	Ax. 1
	$\therefore AD$ is $\parallel$ to $BC.$	§ 345
	$\therefore PA : AB = PD : DC = m : n.$	§ 342

Q. E. F.

Discussion. The problem is impossible when the distance from P to the nearest point of the circumference is greater than $\frac{m}{n}$ of the diameter of the $\odot$.

Ex. 296. *To draw through a given external point P a secant PAB to a given circle so that* $\overline{AB}^2 = PA \times PB$.

Let P be a point exterior to the $\odot ABD$.

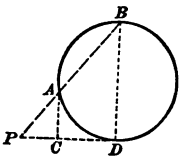
To draw a secant PAB so that $\overline{AB}^2 = PA \times PB$.

Construction. Draw the tangent PD.

Divide PD so that

$$PC : CD = CD : PD. \qquad \S\ 390$$

With P as centre and radius equal to CD, cut the circumference at A. Draw PAB, AC, and BD.



PAB is the secant required.

Proof.

	$PC : PA = PA : PD.$	Const.
But	$PA : PD = PD : PB.$	§ 381
	$\therefore PC : PA = PD : PB.$	Ax. 1
	$\therefore AC$ is $\parallel$ to $BD.$	§ 345
	$\therefore PC : PA = CD : AB = PA : AB.$	§ 342
But	$PC : PA = PA : PD.$	
	$\therefore AB = PD.$	
	$\therefore PA : AB = AB : PB.$	
	$\therefore \overline{AB}^2 = PA \times PB.$	Q. E. F.

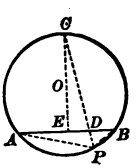
Discussion. The problem is impossible when PD is greater than the diameter of the $\odot$.

Ex. 297. *To find a point P in the arc subtended by a given chord AB so that the ratio $PA : PB$ shall equal the given ratio $m : n$.*

Let AB be a chord of the $\odot ABC$.

To find a point P in the arc AB so that

$$PA : PB = m : n.$$



Construction. Through the centre O draw $EC \perp$ to AB. Take D on AB so that $AD : DB = m : n$. Draw CDP, PA, and PB.

P is the point required.

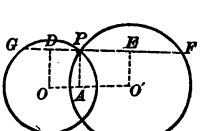
Proof.

arc AC = arc $BC.$	§ 245
$\therefore \angle APC = \angle BPC.$	§ 293
$\therefore AP : PB = AD : DB = m : n.$	§ 348

Q. E. F.

Ex. 298. *To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio $m : n$.*

Let O and O' be the centres of two Ⓢ intersecting at P.



To draw through P a secant so that the two chords formed shall be to each other in the ratio $m : n$.

Construction. Draw the line of centres OO'.

Divide OO' so that

$$OA : AO' = m : n. \qquad \S\ 385$$

Draw AP, and draw $GPF \perp$ to AP.

Then GPF is the secant required.

Proof. Draw OD and $O'E \perp$ to GPF.

$$OA : AO' = DP : PE = m : n. \qquad \S\ 344$$

But $DP = \frac{1}{2} GP$,

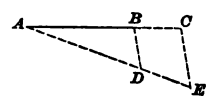
and $PE = \frac{1}{2} PF$. $\S\ 245$

$$\therefore GP : PF = m : n. \qquad \text{Q. E. F.}$$

Ex. 299. *Having given the greater segment of a line divided in extreme and mean ratio, to construct the line.*

Let AB be the greater segment of a line divided in extreme and mean ratio.

To construct the line.



Construction. Draw AE at a convenient angle with AB. Divide AE so that

$$AE : AD = AD : DE. \qquad \S\ 390$$

Find BC, the fourth proportional to AD, AB, and DE.

AC is the line required.

Proof.

$$AC : AB = AE : AD. \qquad \S\ 343$$

But $$AE : AD = AD : DE. \qquad \text{Const.}$$

Also, $$AD : DE = AB : BC. \qquad \text{Const.}$$

$$\therefore AC : AB = AB : BC. \qquad \text{Ax. 1}$$

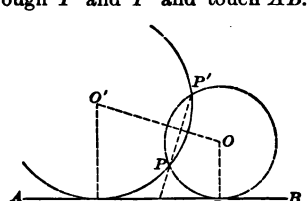
Q. E. F.

Ex. 300. *To construct a circle which shall pass through two given points and touch a given straight line.*

Let P and P' be the two points, and AB the given straight line.

To construct a $\odot$ which shall pass through P and P' and touch AB.

Construction. Draw $P'P$ cutting AB at C.



Find the mean proportional between $P'C$ and PC. $\S\ 388$

Cut off CD equal to this mean proportional.

Erect $OD \perp$ to AB, and draw $O'O$ bisecting $P'P$ at right angles.

At O, the point of intersection of $O'O$ and OD, describe a circumference with a radius OP.

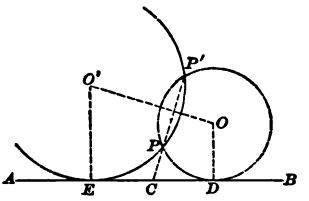
This circumference passes through P and P', and is tangent to AB at D.

Proof. Since O is equally distant from P and P', the circumference passes through P'.

Also, $P'C : CD = CD : PC$. Const.

$\therefore$ CD is tangent to the $\odot$ at D. § 381

Q. E. F.

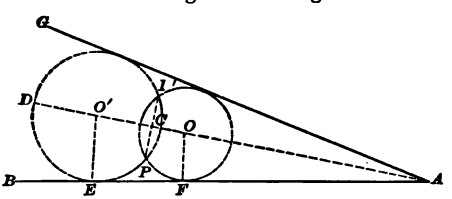


Discussion. By taking CE equal to CD, a second solution may be found.

Ex. 301. *To construct a circle which shall pass through a given point and touch two given straight lines.*

Let AB and AG be two straight lines intersecting at A, and P be the given point.

To pass a circumference through P touching AB and AG.



Construction. Draw AD bisecting the $\angle BAG$.

Draw $PC \perp$ to AD, and prolong it to P', making CP' equal to PC.

Construct a $\odot$ passing through P and P' and tangent to AB. Ex. 300

This $\odot$ is the $\odot$ required.

Proof. Since AD is the $\perp$ bisector of PP', the centre must lie in AD. § 160

Since AD is the bisector of $\angle BAG$, the $\odot$ must touch AG. § 162

Q. E. F.

Discussion. As in Ex. 300, two solutions are possible.

Special Case: when AB and AG are $\parallel$.

Ex. 302. *To inscribe a square in a semicircle.*

Let $EDCF$ be a semicircle.

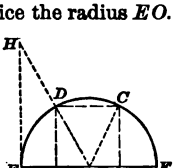
To inscribe a square in $EDCF$.

Construction. At E draw the tangent EH equal to twice the radius EO.

Draw HO, cutting the circumference at D.

Draw $DA \perp$ to EF, $DC \parallel$ to EF, and $CB \perp$ to EF.

$ABCD$ is the square required.



Proof. $ABCD$ is a rectangle by construction.

Draw OC.

The rt. $\triangle$ OAD and OBC are equal. § 151

For $OD = OC$ (§ 217), and $AD = BC$. § 180
$\therefore OA = OB$. § 128
In the $\triangle OEH$, AD is $\parallel$ to EH. § 104
$\therefore OA : AD = OE : EH$.
But $OE = \frac{1}{2} EH$. Const.
$\therefore OA = \frac{1}{2} AD$.
$\therefore OA + OB = AD$.
$\therefore ABCD$ is the square required. Q. E. F.

Ex. 303. *To inscribe a square in a given triangle.*

Let ABC be the given $\triangle$.

To inscribe a square in the $\triangle ABC$.

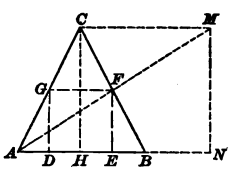
Construction. Draw the altitude CH.

On HB produced construct a square $HNMC$ with CH as one side.

Draw AM, cutting BC at F.

Construct the rectangle $DEFG$.

$DEFG$ is the square required.



Proof. The rt. $\triangle AEF$ and ANM are similar. § 356
$\therefore EF : NM = AF : AM$. § 351
The $\triangle AGF$ and ACM are similar. § 354
$\therefore GF : CM = AF : AM$. § 351
$\therefore EF : NM = GF : CM$. Ax 1.
But $NM = CM$. Const.
$\therefore EF = GF$.
$\therefore DEFG$ is the square required. Q. E. F.

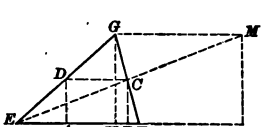
Ex. 304. *To inscribe in a given triangle a rectangle similar to a given rectangle.*

Let EFG be the given $\triangle$, and R the given rectangle.

To inscribe in EFG a rectangle similar to R.

Construction. Draw the altitude GH.

On HF produced construct a rectangle similar to R with GH as one side.



Draw EM, cutting FG at C.

Construct the rectangle $ABCD$.

Then $ABCD$ is the rectangle required.

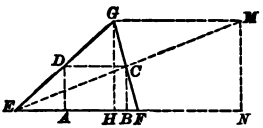
Proof. The rt. $\triangle ABC$ and ANM are similar. § 356
$\therefore BC : NM = EC : EM$. § 351

The $\triangle EDC$ and EGM are similar. § 354

$\therefore DC : GM = EC : EM$. § 351

$\therefore BC : NM = DC : GM$. Ax. 1

$\therefore ABCD$ is similar to $HNMG$, and also to R. Q. E. F.



Ex. 305. *To inscribe in a circle a triangle similar to a given triangle.*

Let $A'B'C'$ be the given $\odot$, and ABC the given $\triangle$.

To inscribe in O a $\triangle$ similar to ABC.

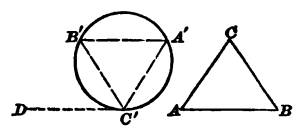
Construction. At any point in the circumference draw the tangent $C'D$.

At C' draw a chord, making $\angle B'C'D$ equal to $\angle A$.

At B' make the $\angle A'B'C'$ equal to $\angle B$.

Draw $A'C'$.

$\triangle A'B'C'$ is the $\triangle$ required.



Proof. $\angle A' = \angle B'C'D$ (being measured by $\frac{1}{2}$ the same arc).

$\therefore \angle A' = \angle A$. Ax. 1

Now $\angle B' = \angle B$. Const.

$\therefore \triangle ABC$ and $A'B'C'$ are similar. § 355

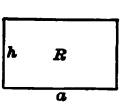
Q. E. F.

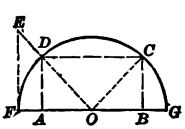
Ex. 306. *To inscribe in a given semicircle a rectangle similar to a given rectangle.*

Let R be the given rectangle, and $FDCG$ the given semicircle.

To inscribe in $FDCG$ a rectangle similar to R.

Construction. Find x, the fourth proportional to $\frac{1}{2}a$, h, and the radius OF. § 386





At F draw the tangent FE equal to x.

Draw OE, cutting the circumference at D.

Construct the rectangle $ABCD$.

Then $ABCD$ is the rectangle required.

Proof. Draw OC.

$OD = OC$. § 217

$AD = BC$. Const.

$\therefore \triangle OAD = \triangle OBC$. § 151

$\therefore OA = OB$. § 128

The rt. $\triangle OAD$ and OFE are similar. § 356

$\therefore OA : AD = OF : FE$. § 351

But $\frac{1}{2}a : h = OF : FE$. Const.

$\therefore OA : AD = \frac{1}{2}a : h$.

But $OA = \frac{1}{2}AB$. $\therefore \frac{1}{2}AB : AD = \frac{1}{2}a : h$.

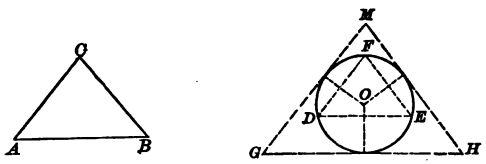
$\therefore ABCD$ is similar to R. Q. E. F.

Ex. 307. *To circumscribe about a circle a triangle similar to a given triangle.*

Let ABC be the given $\triangle$, and O the centre of the given $\odot$.

To circumscribe a $\triangle$ about the $\odot$ similar to the $\triangle ABC$.

Construction. Inscribe in the $\odot$ a $\triangle DEF$ similar to $\triangle ABC$. Ex. 305



Draw radii $\perp$ to the sides of the $\triangle DEF$.

At the extremities of these radii, draw the tangents GH, HM, and MG.

Then the $\triangle GHM$ is the $\triangle$ required.

Proof. GH, HM, MG are $\parallel$ to DE, EF, FD, respectively. §§ 254, 104

$\therefore$ the $\measuredangle$ GHM and DEF are similar. § 359

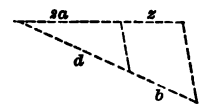
$\therefore$ the $\measuredangle$ GHM and ABC are similar. Q. E. F.

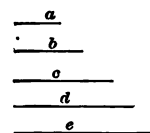
Ex. 308. *To construct the expression,* $x = \dfrac{2\,abc}{de}$; *that is,* $\dfrac{2\,ab}{d} \times \dfrac{c}{e}$.

Given the lines a, b, c, d, and e.

To construct the expression,

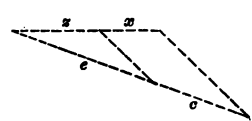
$$x = \frac{2\,abc}{de}.$$





Construction. Find z, the fourth proportional to d, $2\,a$, and b. § 386

Find x, the fourth proportional to e, z, and c.



Then x is the line required.

Proof.

$d : 2\,a = b : z.$ Const.

$$\therefore z = \frac{2\,ab}{d}.$$

$e : z = c : x.$ Const.

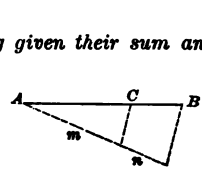
$$\therefore x = \frac{cz}{e}.$$

Substitute for z its equal $\dfrac{2\,ab}{d}$.

Then $$x = \frac{2\,abc}{de}.$$ Q. E. F.

Ex. 309. *To construct two straight lines, having given their sum and their ratio.*

Let AB be the sum of two lines, and $m : n$ their ratio.



To find the lines.

Construction. Divide AB into segments proportional to m and n. § 385

Then AC and CB are the lines required.

Proof. By construction $AC + CB = AB$,

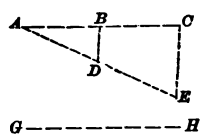
and $AC : CB = m : n$. Q. E. F.

Ex. 310. *To construct two straight lines, having given their difference and their ratio.*

Let a equal the difference of two straight lines, and $m : n$ their ratio.

To construct the lines.

Construction. Take AD equal to $m - n$, AB equal to n, and DE equal to a.



Find BC, the fourth proportional to AD, AB, and DE. § 386

Take GH equal to $BC + a$.

Then BC and GH are the lines required.

Proof. By construction $AD : AB = DE : BC$.

Also, $AD = m - n$,

$AB = n$,

and $DE = GH - BC$.

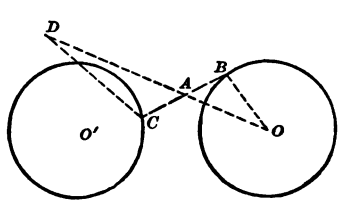
Substituting these values, $$\frac{m-n}{n} = \frac{GH - BC}{BC}$$

or $$\frac{m}{n} - 1 = \frac{GH}{BC} - 1.$$

$$\therefore GH : BC = m : n.$$ Q. E. F.

Ex. 311. *Given two circles, with centres O and O', and a point A in their plane, to draw through the point A a straight line, meeting the circumferences at B and C, so that $AB : AC = m : n$.*

Let O and O' be the centres of the given ⊙, and A the given point.



To draw a line through A, meeting the circumferences at B and C, so that $AB : AC = m : n$.

Construction. Draw OA, and prolong it to D, so that

$OA : AD = m : n$. § 386

Find x, the fourth proportional to m, n, and the radius of the circle O.

With D as centre, and radius x, cut the circumference of O' at C, and draw DC.

Draw $OB \parallel$ to CD, and draw AB and AC.

Then BAC is a straight line and $AB : AC = m : n$.

Proof. In the $\triangle ABO$ and ACD,

$$\angle D = \angle O, \qquad \S\ 110$$

and $$OA : AD = m : n = OB : DC. \qquad \text{Const.}$$

$\therefore$ the $\triangle ABO$ and ACD are similar. $\S\ 357$

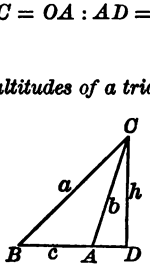
$\therefore \angle OAB = \angle DAC.$ $\S\ 351$

Since OAD is a straight line, BAC is a straight line.

Also, $$AB : AC = OA : AD = m : n. \qquad \S\ 351$$

Q. E. F.

Ex. 312. *To compute the altitudes of a triangle in terms of its sides.*



At least one of the angles A or B is acute. Suppose B is acute.

In the $\triangle CDB$, $\quad h^2 = a^2 - \overline{BD}^2.$ $\S\ 372$

In the $\triangle ABC$, $\quad b^2 = a^2 + c^2 - 2c \times BD.$ $\S\ 375$

Whence $$BD = \frac{a^2 + c^2 - b^2}{2c}.$$

Hence, $$h^2 = a^2 - \frac{(a^2 + c^2 - b^2)^2}{4c^2} = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$$

$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$$

$$= \frac{\{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\}}{4c^2}$$

$$= \frac{(a + b + c)(a + c - b)(b + a - c)(b - a + c)}{4c^2}.$$

Let $$a + b + c = 2s.$$

Then $$a + c - b = 2(s - b),$$

$$b + a - c = 2(s - c),$$

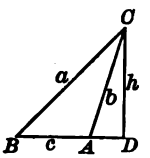
and $$b - a + c = 2(s - a).$$

Hence, $$h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2}.$$

By simplifying, and extracting the square root,

$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

If h' is the altitude on the side a, and h'' is the altitude on the side b,

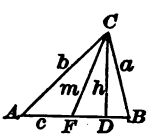


then $$h' = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)},$$

and $$h'' = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 313. *To compute the medians of a triangle in terms of its sides.*

$$a^2 + b^2 = 2m^2 + 2\left(\frac{c}{2}\right)^2. \qquad \S\ 377$$



Whence $$4m^2 = 2(a^2 + b^2) - c^2.$$

$$\therefore m = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$

If m' is the median to the side a, and m'' is the median to the side b, then

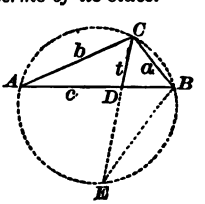
$$m' = \tfrac{1}{2}\sqrt{2(b^2 + c^2) - a^2},$$

and $$m'' = \tfrac{1}{2}\sqrt{2(a^2 + c^2) - b^2}.$$

Ex. 314. *To compute the bisectors of a triangle in terms of its sides.*

$$t^2 = ab - AD \times BD. \qquad \S\ 383$$

$$\frac{AD}{b} = \frac{BD}{a} = \frac{AD + BD}{a + b} = \frac{c}{a + b}. \qquad \S\ 348$$



$$\therefore AD = \frac{bc}{a + b}, \text{ and } BD = \frac{ac}{a + b}.$$

Whence
$$t^2 = ab - \frac{abc^2}{(a + b)^2}$$

$$= ab\left[1 - \frac{c^2}{(a + b)^2}\right]$$

$$= \frac{ab\{(a + b)^2 - c^2\}}{(a + b)^2}$$

$$= \frac{ab(a + b + c)(a + b - c)}{(a + b)^2}$$

$$= \frac{ab \times 2s \times 2(s - c)}{(a + b)^2}.$$

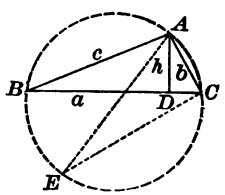
$$\therefore t = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

If t' is the bisector to the side a, and t'' is the bisector to the side b,

then $$t' = \frac{2}{b+c}\sqrt{bcs(s-a)},$$

and $$t'' = \frac{2}{a+c}\sqrt{acs(s-b)}.$$

Ex. 315. *To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.*



$$AC \times AB = AE \times AD, \quad \text{§ 384}$$

or, $$bc = 2R \times AD.$$

But $$AD = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}. \quad \text{Ex. 312}$$

$$\therefore R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. 316. *If the sides of a triangle are* 3, 4, *and* 5, *is the angle opposite* 5 *right, acute, or obtuse?*

A right angle. For $3^2 + 4^2 = 5^2$. § 371

Ex. 317. *If the sides of a triangle are* 7, 9, *and* 12, *is the angle opposite* 12 *right, acute, or obtuse?*

An obtuse angle. For $12^2 > 7^2 + 9^2$. § 376

Ex. 318. *If the sides of a triangle are* 7, 9, *and* 11, *is the angle opposite* 11 *right, acute, or obtuse?*

An acute angle. For $11^2 < 7^2 + 9^2$. § 375

Ex. 319. *The legs of a right triangle are* 8 *inches and* 12 *inches; find the lengths of the projections of these legs upon the hypotenuse, and the distance of the vertex of the right angle from the hypotenuse.*

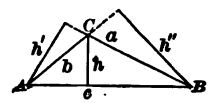
In $\triangle ABC$, $AC = 8$, and $BC = 12$.

To find AD, DB, and DC.

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{8^2 + 12^2}$$
$$= 14.422. \quad \text{§ 371}$$
$$AB : AC = AC : AD, \quad \text{§ 367}$$
$$14.422 : 8 = 8 : AD,$$
$$AD = 4.438.$$
$$DB = AB - AD = 9.984.$$
$$DC = \sqrt{AD \times DB} = \sqrt{4.438 \times 9.984} = 6.656. \quad \text{§ 367}$$

Ex. 320. *If the sides of a triangle are 6 inches, 9 inches, and 12 inches, find the lengths* (1) *of the altitudes;* (2) *of the medians;* (3) *of the bisectors;* (4) *of the radius of the circumscribed circle.*

In $\triangle ABC$, $a = 9$, $b = 6$, $c = 12$.



(1) Let h, h', and h'' be altitudes drawn to c, a, and b, respectively.

To find the lengths of h, h', and h''.

$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$

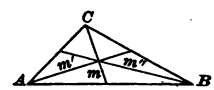
$$= \tfrac{2}{12}\sqrt{\tfrac{27}{2}(\tfrac{27}{2}-9)(\tfrac{27}{2}-6)(\tfrac{27}{2}-12)} \qquad \text{Ex. 312}$$

$$= 4.357.$$

$$h' = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} = \tfrac{2}{9}\sqrt{\tfrac{27}{2}(\tfrac{27}{2}-9)(\tfrac{27}{2}-6)(\tfrac{27}{2}-12)} = 5.809.$$

$$h'' = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} = \tfrac{2}{6}\sqrt{\tfrac{27}{2}(\tfrac{27}{2}-9)(\tfrac{27}{2}-6)(\tfrac{27}{2}-12)} = 8.714.$$

(2) Let m, m', and m'' be medians drawn to c, a, and b, respectively.



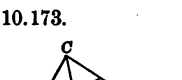
To find the lengths of m, m', and m''.

$$m = \tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2} = \tfrac{1}{2}\sqrt{2\times(9^2+6^2)-12^2} = 4.743. \qquad \text{Ex. 313}$$

$$m' = \tfrac{1}{2}\sqrt{2(b^2+c^2)-a^2} = \tfrac{1}{2}\sqrt{2\times(6^2+12^2)-9^2} = 8.351.$$

$$m'' = \tfrac{1}{2}\sqrt{2(a^2+c^2)-b^2} = \tfrac{1}{2}\sqrt{2\times(9^2+12^2)-6^2} = 10.173.$$

(3) Let t, t', and t'' be the bisectors of $\measuredangle C$, A, and B, respectively.



To find the lengths of t, t', t''.

$$t = \frac{2}{a+b}\sqrt{abs(s-c)} = \frac{2}{9+6}\sqrt{9\times 6\times \tfrac{27}{2}\times(\tfrac{27}{2}-12)} = 4.409. \qquad \text{Ex. 314}$$

$$t' = \frac{2}{b+c}\sqrt{bcs(s-a)} = \frac{2}{6+12}\sqrt{6\times 12\times \tfrac{27}{2}\times(\tfrac{27}{2}-9)} = 7.348.$$

$$t'' = \frac{2}{a+c}\sqrt{acs(s-b)} = \frac{2}{9+12}\sqrt{9\times 12\times \tfrac{27}{2}\times(\tfrac{27}{2}-6)} = 9.959.$$

(4) Let R be the radius of the circumscribed $\odot$.

To find the length of R.

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{Ex. 315}$$

$$= \frac{9\times 6\times 12}{4\sqrt{\tfrac{27}{2}(\tfrac{27}{2}-9)(\tfrac{27}{2}-6)(\tfrac{27}{2}-12)}} = 6.196.$$

Ex. 321. *A line is drawn parallel to a side AB of a triangle ABC, cutting AC in D, BC in E. If AD : DC = 2 : 3, and AB = 20 inches, find DE.*

$AD : DC = 2 : 3.$

By composition, $AD + DC : DC = 2 + 3 : 3.$ § 332

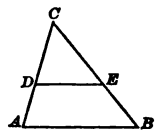
Or $AC : DC = 5 : 3.$

But $AC : DC = AB : DE.$ § 351

$\therefore AB : DE = 5 : 3.$

Or $20 : DE = 5 : 3.$

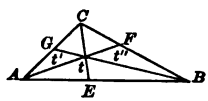
$\therefore DE = 12$ inches.



Ex. 322. *The sides of a triangle are* 9, 12, 15. *Find the segments of the sides made by bisecting the angles.*

In the $\triangle ABC$, let t, t', t'' be the bisectors of $\measuredangle C$, A, and B, respectively.

$AB = 15$, $BC = 12$, $AC = 9$.



To find the lengths of AE, EB, BF, FC, CG, and GA.

$AE : EB = AC : BC = 9 : 12.$ § 348

$\therefore AE = \frac{9}{21}$ of $15 = 6\frac{3}{7}$,

$EB = \frac{12}{21}$ of $15 = 8\frac{4}{7}$.

In like manner, $BF = \frac{15}{24}$ of $12 = 7\frac{1}{2}$,

$FC = \frac{9}{24}$ of $12 = 4\frac{1}{2}$,

$CG = \frac{12}{27}$ of $9 = 4$,

$GA = \frac{15}{27}$ of $9 = 5$.

Ex. 323. *A tree casts a shadow* 90 *feet long, when a post* 6 *feet high casts a shadow* 4 *feet long. How high is the tree?*

Let $x =$ the height of the tree.

Then $4 : 90 = 6 : x.$ $\therefore x = 135$ feet.

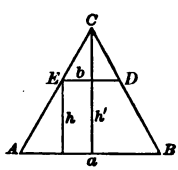
Ex. 324. *The lower and upper bases of a trapezoid are* a, b, *respectively; and the altitude is* h. *Find the altitudes of the two triangles formed by producing the legs until they meet.*

Let ABC be a $\triangle$ formed by producing the legs of the trapezoid $ABDE$, h and h' be the altitudes of $ABDE$ and ABC, respectively, and a and b be the bases of the trapezoid.

To find the altitudes of the $\measuredangle ABC$ and EDC in terms of h, a, and b.

The altitude of $\triangle EDC$ is $h' - h$.

Since AB and ED are $\parallel$,



$h' - h : h' = b : a.$ § 361

$\therefore ah' - ah = bh'.$ § 327

$$\therefore h' = \frac{ah}{a-b},$$

and $$h' - h = \frac{ah}{a-b} - h = \frac{bh}{a-b}.$$

Ex. 325. *The sides of a triangle are* 6, 7, 8, *respectively. In a similar triangle the side homologous to* 8 *is* 40. *Find the other two sides.*

Let x = the side homologous to 6,

y = the side homologous to 7.

Then $6 : 8 = x : 40$, § 351

and $7 : 8 = y : 40$.

$\therefore x = 30$, and $y = 35$.

Ex. 326. *The perimeters of two similar polygons are* 200 *feet and* 300 *feet. If a side of the first is* 24 *feet, find the homologous side of the second.*

Let x = the side of the second homologous to 24.

Then $200 : 300 = 24 : x$. § 364

$\therefore x = 36$ feet.

Ex. 327. *How long a ladder is required to reach a window* 24 *feet high, if the lower end of the ladder is* 10 *feet from the side of the house?*

Let x = the length of the ladder.

Then $x = \sqrt{24^2 + 10^2} = 26$. § 371

26 feet. *Ans.*

Ex. 328. *If the side of an equilateral triangle is a, find the altitude.*

Let x = the length of the altitude.

Then $x = \sqrt{a^2 - (\frac{1}{2}a)^2} = \sqrt{\frac{3}{4}a^2} = \frac{a}{2}\sqrt{3}$. § 372

Ex. 329. *If the altitude of an equilateral triangle is h, find the side.*

Let a be the side of the equilateral $\triangle$.

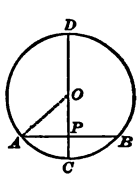
$a^2 - (\frac{1}{2}a)^2 = h^2$. § 372

$\therefore a^2 = \frac{4h^2}{3}$.

$\therefore a = \frac{2h}{3}\sqrt{3}$.

Ex. 330. *Find the length of the longest chord and of the shortest chord that can be drawn through a point* 6 *inches from the centre of a circle whose radius is* 10 *inches.*

Let P be the given point, CD a diameter drawn through P, and AB a chord through $P \perp$ to CD.



Draw AO.

CD is the longest chord and $= 20$ inches. § 252

AB is the shortest chord. Ex. 109

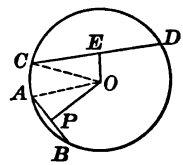
$AP = \sqrt{\overline{AO}^2 - \overline{OP}^2} = \sqrt{10^2 - 6^2} = 8$. § 372

But $AP = \frac{1}{2}AB$. § 245

$\therefore AB = 16$ inches.

Ex. 331. *The distance from the centre of a circle to a chord* 10 *feet long is* 12 *feet. Find the distance from the centre to a chord* 24 *feet long.*

Let AB be 10 feet, CD be 24 feet, and $OP \perp$ to AB be 12 feet.



To find the length of $OE \perp$ to CD.

Draw OA and OC.

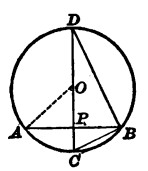
$OA = \sqrt{\overline{OP}^2 + \overline{AP}^2} = \sqrt{12^2 + 5^2} = 13 = OC.$ § 371

$OE = \sqrt{\overline{OC}^2 - \overline{CE}^2} = \sqrt{13^2 - 12^2} = 5.$ § 372

$\therefore CE = 5$ feet.

Ex. 332. *The radius of a circle is* 5 *inches. Through a point* 3 *inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance from one end of the chord to the ends of the diameter.*

Let AB be a chord drawn $\perp$ to the diameter CD and 3 inches from the centre O.



To find the lengths of AB, BD, and BC.

Draw OA.

$AP = \sqrt{\overline{OA}^2 - \overline{OP}^2} = \sqrt{5^2 - 3^2} = 4.$ § 372

$\therefore AB = 2 \times AP = 8$ inches. § 245

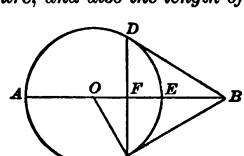
$BD = \sqrt{\overline{PB}^2 + \overline{PD}^2} = \sqrt{4^2 + 8^2} = 8.944.$ § 371

$BC = \sqrt{\overline{PB}^2 + \overline{PC}^2} = \sqrt{4^2 + 2^2} = 4.472.$ § 371

$\therefore BD = 8.944$ inches, and $BC = 4.472$ inches.

Ex. 333. *The radius of a circle is* 6 *inches. Find the lengths of the tangents drawn from a point* 10 *inches from the centre, and also the length of the chord joining the points of contact.*

Let AB be a secant drawn through the centre O, BD and BC be tangents, and DC be a chord joining D and C.



To find the lengths of BC, BD, and DC.

$AB = 16$, $EB = 4$, $OB = 10$.

$\overline{BC}^2 = AB \times EB = 16 \times 4.$ § 381

$\therefore BC = 8$ inches $= BD.$ § 261

$OF : FB = \overline{OC}^2 : \overline{BC}^2 = 36 : 64.$ § 368

$\therefore FB = \frac{64}{100}$ of 10 inches $= 6.4$ inches.

$CF = \sqrt{\overline{BC}^2 - \overline{FB}^2} = \sqrt{8^2 - 6.4^2} = 4.8.$ § 372

$\therefore DC = 2 \times CF = 9.6$ inches. § 245

Ex. 334. *The sides of a triangle are* 407 *feet*, 368 *feet, and* 351 *feet. Find the three bisectors and the three altitudes.*

In $\triangle ABC$, $a = 407$, $b = 368$, $c = 351$.

(1) Let t, t', and t'' be the bisectors of $\measuredangle C$, A, and B, respectively.

To find the lengths of t, t', t''.

$$t = \frac{2}{a+b}\sqrt{abs\,(s-c)} = \frac{2}{407+368}\sqrt{407 \times 368 \times 563 \times 212} = 345.5.$$

Ex. 314

$$t' = \frac{2}{b+c}\sqrt{bcs\,(s-a)} = \frac{2}{368+351}\sqrt{368 \times 351 \times 563 \times 156} = 296.3.$$

$$t'' = \frac{2}{a+c}\sqrt{acs\,(s-b)} = \frac{2}{407+351}\sqrt{407 \times 351 \times 563 \times 195} = 330.4.$$

$\therefore$ the three bisectors are 345.5 feet, 296.3 feet, 330.4 feet.

(2) Let h, h', and h'' be the altitudes drawn to c, a, and b, respectively.

To find the lengths of h, h', h''.

$$h = \frac{2}{c}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \frac{2}{351}\sqrt{563 \times 156 \times 195 \times 212} = 343.3.$$

Ex. 312

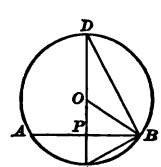
$$h' = \frac{2}{a}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \frac{2}{407}\sqrt{563 \times 156 \times 195 \times 212} = 296.1.$$

$$h'' = \frac{2}{b}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \frac{2}{368}\sqrt{563 \times 156 \times 195 \times 212} = 327.5.$$

$\therefore$ the three altitudes are 343.3 feet, 296.1 feet, 327.5 feet.

Ex. 335. *If a chord* 8 *inches long is* 3 *inches distant from the centre of the circle, find the radius, and the chords drawn from the end of the chord to the ends of the diameter which bisects the chord.*

Let AB be a chord 8 inches long, 3 inches distant from O, and CD be the diameter bisecting AB.



To find the lengths of OB, BD, and BC.

$$OB = \sqrt{\overline{OP}^2 + \overline{PB}^2} = \sqrt{3^2 + 4^2} = 5. \qquad \text{§ 371}$$

$$BD = \sqrt{\overline{DP}^2 + \overline{PB}^2} = \sqrt{8^2 + 4^2} = 8.944.$$

$$BC = \sqrt{\overline{PB}^2 + \overline{PC}^2} = \sqrt{4^2 + 2^2} = 4.472.$$

$\therefore$ $OB = 5$ inches, $BD = 8.944$ inches, $BC = 4.472$ inches.

Ex. 336. *From the end of a tangent* 20 *inches long a secant is drawn through the centre of the circle. If the external segment of this secant is* 8 *inches, find the radius of the circle.*

$CD = 20$ inches, $CB = 8$ inches.

To find the length of OB.

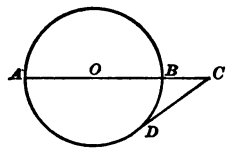
$AC : CD = CD : CB.$ § 381

$AC : 20 = 20 : 8.$

$\therefore AC = 50.$

$AB = AC - CB = 42.$

$OB = \frac{1}{2} AB = 21$ inches.



Ex. 337. *The radius of a circle is* 13 *inches. Through a point* 5 *inches from the centre any chord is drawn. What is the product of the two segments of the chord? What is the length of the shortest chord that can be drawn through the point?*

Let FE be a diameter, AB a chord $\perp$ to FE, and CPD any other chord.

$OP = 5$ inches, $OA = 13$ inches.

To find the value of $CP \times PD$, and the length of AB.

$EP = 8$, and $PF = 18$.

$\therefore CP \times PD = 8 \times 18 = 144.$ § 378

AB is the shortest chord drawn through P. Ex. 109

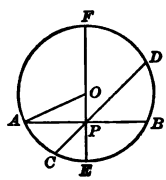
$AP \times PB = 144.$

But $AP = PB.$

$\therefore \overline{AP}^2 = 144.$

$\therefore AP = 12.$

$\therefore AB = 24$ inches.



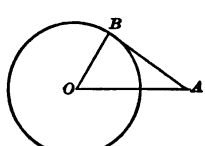
Ex. 338. *The radius of a circle is* 9 *inches, and the length of a tangent* 12 *inches. Find the length of a line drawn from the extremity of the tangent to the centre of the circle.*

$AB = 12$ inches, $OB = 9$ inches.

To find the length of AO.

$AO = \sqrt{\overline{AB}^2 + \overline{OB}^2} = \sqrt{12^2 + 9^2} = 15.$ § 371

$\therefore AO = 15$ inches.



Ex. 339. *Two circles have radii of* 8 *inches and* 3 *inches, respectively, and the distance between their centres is* 15 *inches. Find the lengths of their common tangents.*

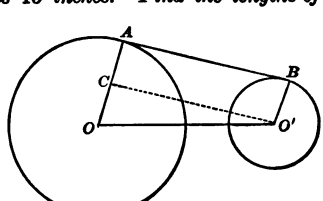
Case I. Let AB be the external tangent.

To find the length of AB.

Draw $O'C \parallel$ to BA.

$OO' = 15,$

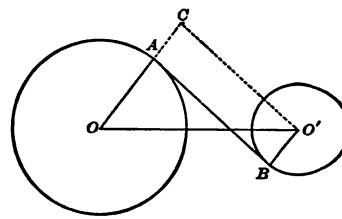
$OC = 5.$



$$O'C = \sqrt{\overline{OO'}^2 - \overline{OC}^2} = \sqrt{15^2 - 5^2} = 14.142. \qquad \text{§ 372}$$

$$AB = O'C = 14.142 \text{ inches.}$$

CASE II. Let AB be the internal tangent.



To find the length of AB.

Make OC equal to $OA + O'B$.

Draw $O'C$.

$OC = 11$,

$OO' = 15$.

$$O'C = \sqrt{\overline{OO'}^2 - \overline{OC}^2} = \sqrt{15^2 - 11^2} = 10.198. \qquad \text{§ 372}$$

$$AB = O'C = 10.198 \text{ inches.}$$

EX. 340. *Find the segments of a line* 10 *inches long divided in extreme and mean ratio.*

Let x = the larger segment.

Then $10 - x$ = the smaller segment.

$\therefore 10 : x = x : 10 - x.$

$x^2 = 100 - 10x.$

Transpose, $x^2 + 10x = 100.$

Complete the square, $x^2 + 10x + 25 = 125.$

Extract the square root, $x + 5 = \pm\sqrt{125}.$

$\therefore x = 6.18.$

$\therefore 10 - x = 3.82.$

6.18 inches and 3.82 inches. *Ans.*

EX. 341. *The sides of a triangle are* 4, 5, 5. *Is the largest angle acute, right, or obtuse?*

The $\triangle$ is isosceles, and the largest angle is either of the equal base angles.

$\therefore$ the largest angle is acute.

EX. 342. *Find the third proportional to two lines whose lengths are* 28 *feet and* 42 *feet.*

Let x = the length in feet of the third proportional.

Then $28 : 42 = 42 : x.$

$2 : 3 = 42 : x.$

$\therefore 2x = 126.$

$\therefore x = 63.$ 63 feet. *Ans.*

Ex. 343. *If the sides of a triangle are a, b, b, respectively, find the lengths of the three altitudes.*

$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{Ex. 312}$$

$$= \frac{2}{b}\sqrt{s(s-a)(s-b)(s-b)}$$

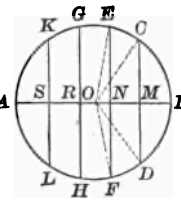
$$= \frac{2(s-b)}{b}\sqrt{s(s-a)}.$$

$$h'' = h = \frac{2(s-b)}{b}\sqrt{s(s-a)}.$$

$$h' = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-b)}$$

$$= \frac{2(s-b)}{a}\sqrt{s(s-a)}.$$

Ex. 344. *The diameter of a circle is* 30 *feet and is divided into five equal parts. Find the lengths of the chords drawn through the points of division perpendicular to the diameter.*



Let the diameter AB be divided into five equal parts at the points S, R, N, M. Through these points of division let the chords KL, GH, EF, CD be drawn $\perp$ to AB.

To find the lengths of these chords.

$OS = OM = 9$, and $OR = ON = 3$.

Draw the radii, OC, OD, OE, OF.

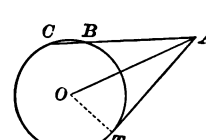
$MD = \sqrt{15^2 - 9^2} = \sqrt{144} = 12.$ § 372

$NF = \sqrt{15^2 - 3^2} = \sqrt{216} = 14.7.$

$\therefore CD = KL = 2\,MD = 2 \times 12$ feet $= 24$ feet,

and $EF = GH = 2\,NF = 2 \times 14.7$ feet $= 29.4$ feet. *Ans.*

Ex. 345. *The radius of a circle is* 2 *inches. From a point* 4 *inches from the centre a secant is drawn so that the internal segment is* 1 *inch. Find the length of the secant.*



Let O be the centre of a circle 2 inches in diameter, A a point 4 inches from O, and ABC a secant so drawn from A that BC is 1 inch in length.

To find the length of AC.

From A draw the tangent AT and draw OT.

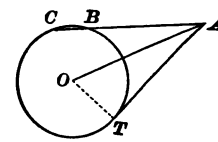
OT is $\perp$ to AT. § 254

$$\therefore AT = \sqrt{\overline{AO}^2 - \overline{OT}^2} = \sqrt{4^2 - 2^2} = \sqrt{12}. \qquad \text{§ 372}$$

Now $AB : AT = AT : AC$, § 381

and $AC = AB + 1$. Hyp.

$$\therefore AB : \sqrt{12} = \sqrt{12} : AB + 1.$$
$$\overline{AB}^2 + AB = 12.$$
$$4\,\overline{AB}^2 + 4\,AB + 1 = 49.$$
$$2\,AB + 1 = 7.$$
$$2\,AB = 6.$$
$$AB = 3.$$
$$AC = AB + 1 = 4.$$



4 inches. *Ans.*

Ex. 346. *The sides of a triangular pasture are* 1551 *yards*, 2068 *yards*, 2585 *yards. Find the median to the longest side.*

In the $\triangle ABC$, $a = 1551$, $b = 2068$, $c = 2585$.

Since $1551^2 + 2068^2 = 2585^2$, the triangle is a rt. $\triangle$, and the median to the hypotenuse is equal to half the hypotenuse. Ex. 21

Therefore, the length of the median required is

$\frac{1}{2}$ of 2585 yards $= 1292\frac{1}{2}$ yards. *Ans.*

Ex. 347. *The diagonal of a rectangle is d, and the perimeter is p. Find the sides.*

Let $x =$ the length of the rectangle.

Then $\frac{p}{2} - x =$ the breadth of the rectangle.

$$\therefore d^2 = x^2 + \left(\frac{p}{2} - x\right)^2. \qquad \text{§ 372}$$
$$d^2 = x^2 + \frac{p^2}{4} - px + x^2.$$
$$2x^2 - px = d^2 - \frac{p^2}{4}.$$
$$16x^2 - (\ ) + p^2 = 8d^2 - p^2.$$
$$4x - p = \pm\sqrt{8d^2 - p^2}.$$
$$4x = p \pm \sqrt{8d^2 - p^2}.$$
$$x = \tfrac{1}{4}\left(p \pm \sqrt{8d^2 - p^2}\right).$$
$$\frac{p}{2} - x = \frac{p}{2} \mp \tfrac{1}{4}\sqrt{8d^2 - p^2}.$$

Ex. 348. *The radius of a circle is r. Find the length of a chord whose distance from the centre is* $\frac{1}{2}\,r$.

Let $x =$ the length of half the chord.

Then $x^2 = r^2 - (\frac{1}{2}\,r)^2 = \frac{3}{4}\,r^2$. § 372

$$\therefore x = \frac{r}{2}\sqrt{3}.$$
$$\therefore 2x = r\sqrt{3}.$$

Ex. 349. *Find the ratio of a rectangular lawn* 72 *yards by* 49 *yards to a grass turf* 18 *inches by* 14 *inches.*

18 inches $= \frac{1}{2}$ yard, 14 inches $= \frac{7}{18}$ yard.

Area of lawn : area of turf $= 72 \times 49 : \frac{1}{2} \times \frac{7}{18}$ § 397

$= 18{,}144 : 1.$

Ex. 350. *Find the ratio of a rectangular courtyard* $18\frac{1}{2}$ *yards by* $15\frac{1}{2}$ *yards to a flagstone* 31 *inches by* 18 *inches.*

31 inches $= \frac{31}{36}$ yard, 18 inches $= \frac{1}{2}$ yard.

Area of courtyard : area of flagstone $= 18\frac{1}{2} \times 15\frac{1}{2} : \frac{31}{36} \times \frac{1}{2}$ § 397

$= 666 : 1.$

Ex. 351. *A square and a rectangle have the same perimeter,* 100 *yards. The length of the rectangle is* 4 *times its breadth. Compare their areas.*

Let $x =$ the breadth of the rectangle.

Then $4x =$ the length of the rectangle,

and $10x =$ the perimeter of the rectangle.

But $100 =$ the perimeter of the rectangle.

$\therefore 10x = 100.$

$\therefore x = 10.$

$4x = 40.$

The side of the square $= \frac{1}{4}$ of $100 = 25.$

Area of rectangle : area of square $= 40 \times 10 : 25 \times 25$ § 397

$= 16 : 25.$

Ex. 352. *On a certain map the linear scale is* 1 *inch to* 5 *miles. How many acres are represented on this map by a square the perimeter of which is* 1 *inch.*

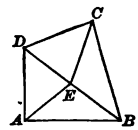
The side of the square is $\frac{1}{4}$ inch, and represents $\frac{5}{4}$ miles.

The area $= (\frac{5}{4} \times \frac{5}{4})$ square miles $= \frac{25}{16}$ square miles.

$\frac{25}{16}$ square miles $= \frac{25}{16} \times 640$ acres $= 1000$ acres.

Ex. 353. *The lines which join the middle point of either diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.*

In the quadrilateral $ABCD$, let E be the middle point of the diagonal BD, and let AE and CE be drawn.



To prove that

the quadrilateral $AECD \doteq$ the quadrilateral $ABCE$.

Proof. $\triangle ABE \doteq \triangle AED$,

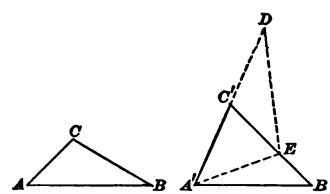
and $\triangle BEC \doteq \triangle CED$. § 404

$\therefore \triangle ABE + \triangle BEC \doteq \triangle AED + \triangle CED$. Ax. 2

That is, the quadrilateral $AECD \doteq$ the quadrilateral $ABCE$. Q. E. D.

Ex. 354. *The areas of two triangles which have an angle of the one supplementary to an angle of the other are to each other as the products of the sides including the supplementary angles.*

Let the $\triangle ABC$ and $A'B'C'$ have the $\angle ACB$ and $A'C'B'$ supplements of each other.



To prove that $\triangle ABC : \triangle A'B'C' = CA \times CB : C'A' \times C'B'$.

Proof. Place $\triangle ABC$ in the position $C'DE$, so that $\angle ACB$ and $A'C'B'$ are adjacent, and draw $A'E$.

$A'C'D$ is a straight line. § 90

Now $\dfrac{\triangle C'DE}{\triangle A'C'E} = \dfrac{C'D}{C'A'}$, and $\dfrac{\triangle A'C'E}{\triangle A'B'C'} = \dfrac{C'E}{C'B'}$. § 405

Multiplying these equalities,

$$\frac{\triangle C'DE}{\triangle A'B'C'} = \frac{C'D \times C'E}{C'A' \times C'B'}.$$

Substituting, $\triangle ABC : \triangle A'B'C' = CA \times CB : C'A' \times C'B'$. Q. E. D.

Ex. 355. *Prove this proposition by* § 410.

The areas of two similar triangles are to each other as the squares of any two homologous sides.

Let ACB and $A'C'B'$ be similar $\triangle$.

To prove that $\dfrac{\triangle ACB}{\triangle A'C'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Proof. Now $\angle A = \angle A'$. § 351

$$\therefore \frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB \times AC}{A'B' \times A'C'} = \frac{AB}{A'B'} \times \frac{AC}{A'C'}.$$ § 410

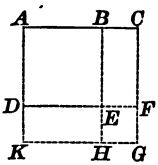
But $$\frac{AC}{A'C'} = \frac{AB}{A'B'}.$$ § 351

$$\therefore \frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$ Q. E. D.

Ex. 356. *The square constructed upon the sum of two straight lines is equivalent to the sum of the squares constructed upon these two lines, increased by twice the rectangle of these lines.*

Let AB and BC be the two straight lines, and AC their sum.

Construct the squares $ACGK$ and $ABED$ upon AC and AB, respectively.



Prolong BE and DE until they meet KG and CG at H and F, respectively.

Now AK, BH, and CG are parallel, and AC, DF, and KG are parallel. § 104

$\therefore$ the sides of the square $EFGH$ are each equal to BC; and the sides of the rectangles $DEHK$ and $BCFE$ are equal to AB and BC. § 180

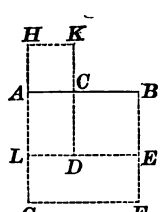
$\therefore$ square $ACGK =$ square $EFGH$ + rectangle $BCFE$ + rectangle $DEHK$.

That is, the square constructed upon AC is equivalent to the sum of the squares constructed upon AB and BC, increased by twice the rectangle of AB and AC.

Ex. 357. *The square constructed upon the difference of two straight lines is equivalent to the sum of the squares constructed upon these two lines, diminished by twice the rectangle of these lines.*

Let AB and AC be the two straight lines, and BC their difference.

Construct the square $ABFG$ upon AB, the square $ACKH$ upon AC, and the square $BEDC$ upon BC (as shown in the figure).



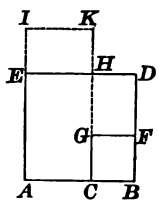
Prolong ED to meet AG in L.

The dimensions of the rectangles $LEFG$ and $HKDL$ are AB and AC, and the square $BCDE$ is evidently the difference between the whole figure and the sum of these rectangles; that is, the square constructed upon BC is equivalent to the sum of the squares constructed upon AB and AC, diminished by twice the rectangle of AB and AC. Q. E. D.

Ex. 358. *The difference between the squares constructed upon two straight lines is equivalent to the rectangle of the sum and difference of these lines.*

Let $ABDE$ and $BCGF$ be the squares constructed upon the two straight lines AB and BC.

The difference between these squares is the polygon $ACGFDE$, which is composed of the rectangles $ACHE$ and $GFDH$.



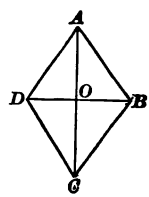
Prolong AE and CH to I and K, respectively, making EI and HK each equal to BC, and draw IK.

The rectangles $GFDH$ and $EHKI$ are equal.

The difference between the squares $ABDE$ and $BCGF$ is then equivalent to the rectangle $ACKI$, which has for dimensions AI, equal to $AB + BC$, and EH, equal to $AB - BC$.

Q. E. D.

Ex. 359. *The area of a rhombus is equal to half the product of its diagonals.*



Let $ABCD$ be a rhombus, and let the diagonals AC and BD intersect at O.

To prove that the area of $ABCD = \frac{1}{2}(AC \times BD)$.

Proof. BD is $\perp$ to AC. Ex. 71

Now area of $\triangle ABC = \frac{1}{2} BO \times AC$, § 403

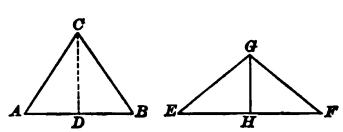
and area of $\triangle ADC = \frac{1}{2} DO \times AC$.

$\therefore \triangle ABC + \triangle ADC$

$= \frac{1}{2} AC \times (BO + DO) = \frac{1}{2} AC \times BD$. Ax. 2

That is, area of rhombus $ABCD = \frac{1}{2} AC \times BD$. Q. E. D.

Ex. 360. *Two isosceles triangles are equivalent if their legs are equal each to each, and the altitude of one is equal to half the base of the other.*



In the isosceles $\triangle ABC$ and EFG, let the legs AC and BC equal the legs EG and FG, and let GH, the altitude of the $\triangle EFG$, equal $\frac{1}{2} AB$.

To prove that $\triangle ABC \cong \triangle EFG$.

Proof. Draw $CD \perp$ to AB.

Then $AD = \frac{1}{2} AB$. § 149

The rt. $\triangle ADC$ and GHE are equal. § 151

For $AC = EG$ and $GH = AD$. Hyp.

Similarly, $\triangle BDC = \triangle GHF$.

$\therefore \triangle ADC + \triangle BDC \cong \triangle GHE + \triangle GHF$. Ax. 2

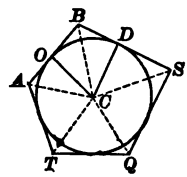
That is, $\triangle ABC \cong \triangle EFG$. Q. E. D.

Ex. 361. *The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.*

Let P represent the perimeter of the circumscribed polygon $ABSQT$, and OC be the radius of the inscribed circle.

To prove that the area of the polygon

$$ABSQT = \tfrac{1}{2}(P \times OC).$$



Proof. Draw CA, CB, CS, CQ, and CT.

The area of the $\triangle ACB = \tfrac{1}{2}(AB \times OC)$. § 403

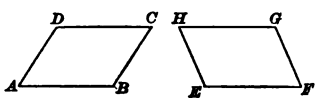
Similarly, the area of each $\triangle$ equals half the product of OC by its base.

Therefore, the area of the polygon, which equals the sum of the areas of the $\triangle$s, is equal to $\tfrac{1}{2}(P \times OC)$.

Q. E. D.

Ex. 362. *Two parallelograms are equal if two adjacent sides of the one are equal, respectively, to two adjacent sides of the other, and the included angles are supplementary.*

Let $ABCD$ and $EFGH$ be two $\square$s, with AB and AD equal, respectively, to EF and EH, and the $\angle$s A and E supplementary.



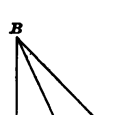
To prove that $\square ABCD = \square EFGH$.

Proof.	$\angle F + \angle E = 2$ rt. $\angle$s.	§ 115
But	$\angle A + \angle E = 2$ rt. $\angle$s.	Hyp.
	$\therefore \angle F = \angle A$.	§ 85
Also,	$FG = EH$.	§ 178
But	$EH = AD$.	Hyp.
	$\therefore FG = AD$.	Ax. 1
	$\therefore \square ABCD = \square EFGH$.	§ 185

Q. E. D.

Ex. 363. *If ABC is a right triangle, C the vertex of the right angle, BD a line cutting AC in D, then*

$$\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2.$$



In the rt. $\triangle ABC$, let BD be drawn, cutting AC in D.

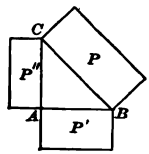
To prove that $\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2$.

Proof.	$\overline{BD}^2 = \overline{BC}^2 + \overline{DC}^2$,	§ 371
and	$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2$.	§ 372
	$\therefore \overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2$.	Ax. 2

Q. E. D.

Ex. 364. *Upon the sides of a right triangle as homologous sides three similar polygons are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the legs.*

Let ABC be a rt. $\triangle$, with $\angle A$ the rt. $\angle$, and let P, P', and P'' be three similar polygons constructed upon BC, AB, and AC as homologous sides.



To prove that $P \simeq P' + P''$.

Proof.

$$\frac{P'}{P} = \frac{\overline{AB}^2}{\overline{BC}^2},$$ § 412

and

$$\frac{P''}{P} = \frac{\overline{AC}^2}{\overline{BC}^2}.$$ § 412

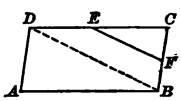
$$\frac{P' + P''}{P} = \frac{\overline{AB}^2 + \overline{AC}^2}{\overline{BC}^2}.$$ Ax. 2

But $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$. § 371

$\therefore P \simeq P' + P''$. Q. E. D.

Ex. 365. *If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one eighth of the parallelogram.*

Let $ABCD$ be a $\square$, and let EF join the middle points of CD and BC.



To prove that $\triangle EFC \simeq \frac{1}{8} \square ABCD$.

Proof. Draw the diagonal BD.

$\triangle EFC : \triangle DBC = EC \times FC : DC \times BC$. § 410

But $EC = \frac{1}{2} DC$, and $BF = \frac{1}{2} BC$. Hyp.

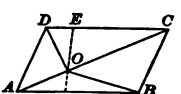
$\therefore \triangle EFC \simeq \frac{1}{2} \triangle DBC$.

But $\triangle DBC \simeq \frac{1}{4} \square ABCD$. § 179

$\therefore \triangle EFC \simeq \frac{1}{8} \square ABCD$. Q. E. D.

Ex. 366. *If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to half the parallelogram.*

Let O be any point within the $\square ABCD$, and let OA, OB, OC, and OD be drawn.



To prove that $\triangle AOB + \triangle DOC \simeq \frac{1}{2} \square ABCD$.

Proof. Through O draw $EF \perp$ to AB.

The area of $\triangle AOB = \frac{1}{2}(AB \times OF)$,

and the area of $\triangle DOC = \frac{1}{2}(DC \times OE)$. § 403

Add these equalities, substituting AB for its equal DC. § 178

Then the area of $\triangle AOB$ + the area of $\triangle DOC = \frac{1}{2}(AB \times EF)$.

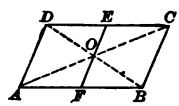
But the area of $\square ABCD = AB \times EF$. § 400

$\therefore \triangle AOB + \triangle DOC \simeq \frac{1}{2} \square ABCD$.

In like manner, $\triangle AOD + \triangle BOC \simeq \frac{1}{2} \square ABCD$. Q. E. D.

Ex. 367. *Every straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equal parts.*

Let $ABCD$ be the given $\square$, O the point of intersection of the diagonals, and EF any line drawn through O.



To prove that EF divides the $\square\ ABCD$ into two equal parts.

Proof. The $\triangle\ AOF$ and COE are equal. § 139

For $AO = CO$. § 184

$\angle OAF = \angle OCE$, § 110

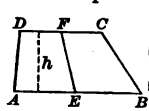
and $\angle AOF = \angle COE$. § 93

In like manner, $\triangle\ FOB = \triangle\ EOD$ and $\triangle\ AOD = \triangle\ COB$.

Therefore, the quadrilateral $AFED$ = the quadrilateral $FBCE$ (being composed of equal $\triangle$ similarly placed). Q. E. D.

Ex. 368. *The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.*

Let $ABCD$ be a trapezoid, and EF the line joining the middle points of the bases AB and CD.



To prove that the trapezoid $AEFD \simeq$ the trapezoid $EBCF$.

Proof. Draw h, the altitude of the trapezoid $ABCD$.

The area of the trapezoid $AEFD = \frac{1}{2}h\,(AE + DF)$,

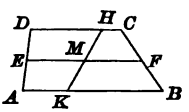
and the area of the trapezoid $EBCF = \frac{1}{2}h\,(EB + FC)$. § 407

But $AE + DF = EB + FC$. Hyp.

$\therefore$ the trapezoid $AEFD \simeq$ the trapezoid $EBCF$. Q. E. D.

Ex. 369. *Every straight line drawn through the middle point of the median of a trapezoid cutting both bases divides the trapezoid into two equivalent parts.*

Let EF be the median of the trapezoid $ABCD$, and M be its middle point. Let HK be a straight line drawn through M cutting DC and AB at H and K, respectively.



To prove that trapezoid $AKHD \simeq$ trapezoid $KBCH$.

Proof. Let h represent the altitude of the trapezoid $ABCD$.

Now $MH = MK$. § 187

Area of trapezoid $AKHD = EM \times h$, § 408

and area of trapezoid $KBCH = MF \times h$.

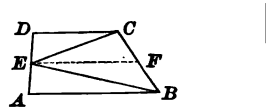
But $EM = MF$. Hyp.

$\therefore$ trapezoid $AKHD \simeq$ trapezoid $KBCH$. Ax. 1

Q. E. D.

Ex. 370. *If two straight lines are drawn from the middle point of either leg of a trapezoid to the opposite vertices, the triangle thus formed is equivalent to half the trapezoid.*

Let $ABCD$ be a trapezoid having $AB \parallel$ to DC, and let EC and EB be lines drawn from E, the middle point of AD, to the opposite vertices.



To prove that $\triangle ECB \doteq \frac{1}{4}$ trapezoid $ABCD$.

Proof. Let $2h$ represent the altitude of the trapezoid.

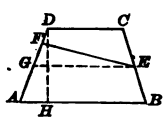
Draw the median EF.

Now	area of $\triangle EFB = \frac{1}{2}h \times EF$,	§ 403
and	area of $\triangle EFC = \frac{1}{2}h \times EF$.	
	$\therefore$ area of $\triangle ECB = h \times EF$.	
But	area of trapezoid $ABCD = 2h \times EF$.	§ 408
	$\therefore \triangle ECB \doteq \frac{1}{4}$ trapezoid $ABCD$.	Q. E. D.

Ex. 371. *The area of a trapezoid is equal to the product of one of the legs and the distance from this leg to the middle point of the other leg.*

In the trapezoid $ABCD$, let EF be drawn from the middle point of $BC \perp$ to AD.

To prove that the area of the trapezoid $ABCD = AD \times EF$.



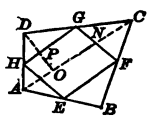
Proof. Draw the median EG and the altitude DH.

The rt. $\triangle ADH$ and GEF are similar.		§ 356
For	$\angle EGF = \angle A$.	§ 112
	$\therefore AD : EG = DH : EF$.	§ 351
	$\therefore AD \times EF = EG \times DH$.	§ 327
But	the area of the trapezoid $ABCD = EG \times DH$.	§ 408
	$\therefore$ the area of the trapezoid $ABCD = AD \times EF$.	Ax. 1
		Q. E. D.

Ex. 372. *The figure whose vertices are the middle points of the sides of any quadrilateral is equivalent to half the quadrilateral.*

In the quadrilateral $ABCD$, let EF, FG, GH, and HE join the middle points of AB, BC, CD, and DA.

To prove that $EFGH \doteq \frac{1}{2}$ the quadrilateral $ABCD$.



Proof. Draw the diagonal AC, and $DO \perp$ to AC.

HG and EF are $\parallel$ to AC and equal to $\frac{1}{2}AC$. § 189

In like manner, GF and HE are $\parallel$.

$\therefore HN$ is a $\square$. § 166

The area of $\square HN = HG \times PO$. § 400

The area of $\triangle ADC = \frac{1}{2}(AC \times DO)$. § 403

But $HG = \frac{1}{2}AC$ and $PO = \frac{1}{2}DO$. § 188

$\therefore \square HN = \frac{1}{2}\triangle ADC$.

In like manner, $\square EN = \frac{1}{2}\triangle ABC$.

$\therefore \square HN + \square EN = \frac{1}{2}(\triangle ADC + \triangle ABC)$. Ax. 2

That is, $\square EFGH = \frac{1}{2}$ the quadrilateral $ABCD$. Q. E. D.

Ex. 373. *If the perimeter of a rectangle is 72 feet, and the length is equal to twice the width, find the area.*

Let x = the breadth.

Then $2x$ = the length,

and $6x$ = the perimeter.

But 72 = the perimeter.

$\therefore 6x = 72$.

$x = 12$.

$2x = 24$.

Area = (24 × 12) square feet = 288 square feet.

Ex. 374. *How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 120 feet long and 36 feet wide?*

The area of the path on the longer side = (136 × 8) square feet = 1088 square feet.

The area of the path on the shorter side = (36 × 8) square feet = 288 square feet.

Area of the path = 2 × (1088 + 288) square feet = 2752 square feet.

9 inches = $\frac{3}{4}$ foot; 4 inches = $\frac{1}{3}$ foot.

Area of a tile = $(\frac{3}{4} \times \frac{1}{3})$ square feet = $\frac{1}{4}$ square foot.

Therefore, the number of tiles is $2752 \div \frac{1}{4} = 11{,}008$.

Ex. 375. *The bases of a trapezoid are 16 feet and 10 feet; each leg is equal to 5 feet. Find the area of the trapezoid.*

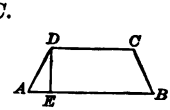
Let $ABCD$ be a trapezoid, and DE be its altitude.

$AB = 16$, $CD = 10$, $AD = 5 = BC$.

Since $AD = BC$,

$AE = \frac{1}{2}(AB - CD) = \frac{1}{2}(16 - 10) = 3$.

$\therefore DE = \sqrt{\overline{AD}^2 - \overline{AE}^2} = \sqrt{5^2 - 3^2} = 4$. § 372



Area of the trapezoid $ABCD = DE \times \frac{1}{2}(AB + CD)$ § 407

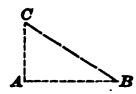
$= 4 \times \frac{1}{2}(16 + 10)$

$= 52$. 52 square feet. *Ans.*

Ex. 376. *Construct a square equivalent to the sum of two squares whose sides are 3 inches and 4 inches.*

Construct the rt. $\angle A$.

Take AB equal to 4, and AC equal to 3. Draw BC.



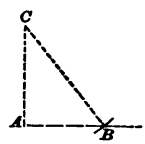
BC is a side of the square required. § 417

$$BC = \sqrt{\overline{AB}^2 + \overline{AC}^2} = \sqrt{4^2 + 3^2} = 5. \quad \text{§ 371}$$

Ex. 377. *Construct a square equivalent to the difference of two squares whose sides are* $2\frac{1}{2}$ *inches and* 2 *inches.*

Construct the rt. $\angle A$. Take AC equal to 2.

With C as centre, and radius equal to $2\frac{1}{2}$, describe an arc cutting AB at B. Draw BC.



AB is a side of the square required. § 418

$$AB = \sqrt{\overline{BC}^2 - \overline{AC}^2} = \sqrt{(\tfrac{5}{2})^2 - 2^2} = 1\tfrac{1}{2}. \quad \text{§ 372}$$

Ex. 378. *Find the side of a square equivalent to the sum of two squares whose sides are* 24 *feet and* 32 *feet.*

Let x = a side of the square required.

$$x = \sqrt{24^2 + 32^2} = 40. \quad \text{§ 417}$$

40 feet. *Ans.*

Ex. 379. *Find the side of a square equivalent to the difference of two squares whose sides are* 24 *feet and* 40 *feet.*

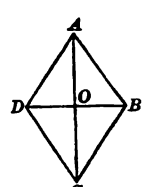
Let x = a side of the square required.

$$x = \sqrt{40^2 - 24^2} = 32. \quad \text{§ 418}$$

32 feet. *Ans.*

Ex. 380. *A rhombus contains* 100 *square feet, and the length of one diagonal is* 10 *feet. Find the length of the other diagonal.*

Let $ABCD$ be a rhombus containing 100 square feet, and let BD be 10 feet.



Area rhombus $ABCD = \frac{1}{2}(AC \times BD)$. Ex. 359

Substituting, $100 = \frac{1}{2}(AC \times 10)$.

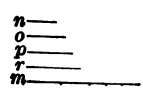
$\therefore AC = 20$.

20 feet. *Ans.*

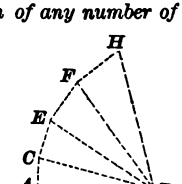
Ex. 381. *To construct a square equivalent to the sum of any number of given squares.*

Let m, n, o, p, r be the sides of the given squares.

To construct a square equivalent to $m^2 + n^2 + o^2 + p^2 + r^2$.



Construction. Take AB equal to m,



Draw AC equal to $n \perp$ to AB at A, and draw BC.
Draw CE equal to $o \perp$ to BC at C, and draw BE.
Draw EF equal to $p \perp$ to BE at E, and draw BF.
Draw FH equal to $r \perp$ to BF at F, and draw BH.
The square constructed on BH is the square required.

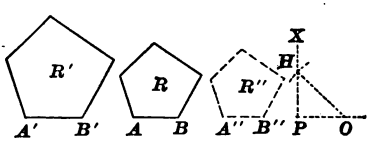
Proof.
$$\overline{BH}^2 = \overline{FH}^2 + \overline{BF}^2 \qquad \text{§ 415}$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{EB}^2$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{CE}^2 + \overline{CB}^2$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{CE}^2 + \overline{AC}^2 + \overline{AB}^2.$$

That is, $\overline{BH}^2 = m^2 + n^2 + o^2 + p^2 + r^2$. Q. E. F.

Ex. 382. *To construct a polygon similar to two given similar polygons and equivalent to their difference.*

Let R and R' be two similar polygons, and AB and $A'B'$ two homologous sides.

To construct a similar polygon equivalent to $R' - R$.



Construction. Construct the rt. $\angle XPO$, and take PO equal to AB.

From O as a centre, with a radius equal to $A'B'$, describe an arc cutting PX at H, and draw OH.

Take $A''B''$ equal to PH, and on $A''B''$, homologous to AB, construct R'' similar to R.

Then R'' is the polygon required.

Proof.
$$\overline{PH}^2 = \overline{OH}^2 - \overline{OP}^2. \qquad \text{§ 416}$$
$$\therefore \overline{A''B''}^2 = \overline{A'B'}^2 - \overline{AB}^2.$$

Now
$$\frac{R'}{R''} = \frac{\overline{A'B'}^2}{\overline{A''B''}^2}, \qquad \text{§ 412}$$

and
$$\frac{R}{R''} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

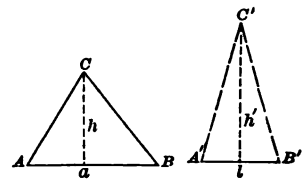
$$\therefore \frac{R' - R}{R''} = \frac{\overline{A'B'}^2 - \overline{AB}^2}{\overline{A''B''}^2} = \frac{\overline{A''B''}^2}{\overline{A''B''}^2} = 1. \qquad \text{Ax. 3}$$

$$\therefore R'' = R' - R. \qquad \text{Q. E. F.}$$

Ex. 383. *To construct a triangle equivalent to a given triangle, and having one side equal to a given length l.*

Let ABC be the given $\triangle$, a its base, and l the given line.

To construct a $\triangle$ on l as a base equivalent to the $\triangle ABC$.



Construction. Draw h, the altitude of the $\triangle ABC$.

Find h', the fourth proportional to l, a, and h. § 386

On l as a base, with an altitude h', construct a $\triangle A'B'C'$.

Then the $\triangle A'B'C'$ is the $\triangle$ required.

Proof. $l : a = h : h'$. Const.

$\therefore l \times h' = a \times h$. § 327

But $l \times h'$ = twice the area of the $\triangle A'B'C'$,

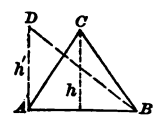
and $a \times h$ = twice the area of the $\triangle ABC$. § 403

$\therefore \triangle A'B'C' = \triangle ABC$. Q. E. F.

Ex. 384. *To transform a triangle into an equivalent right triangle.*

Let ABC be the given $\triangle$.

To transform the $\triangle ABC$ into an equivalent right $\triangle$.



Construction. Draw h, the altitude of the $\triangle ABC$.

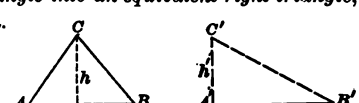
At A erect a $\perp AD$ equal to h. Draw DB.

Then the $\triangle ABD$ is the $\triangle$ required.

Proof. $\triangle ABD = \triangle ABC$. § 404

Q. E. F.

Ex. 385. *To transform a given triangle into an equivalent right triangle, having one leg equal to a given length.*



Let ABC be the given $\triangle$, a its base, and a' the given length.

To transform the $\triangle ABC$ into an equivalent rt. $\triangle$ having one leg equal to a'.

Construction. Draw h, the altitude of the $\triangle ABC$.

Find h', the fourth proportional to a', a, and h. § 386

Take $A'B'$ equal to a'.

At A' erect a $\perp A'C'$ equal to h', and draw $B'C'$.

Then the $\triangle A'B'C'$ is the rt. $\triangle$ required.

Proof. $a' : a = h : h'$. Const.

$\therefore a' \times h' = a \times h$. § 327

But $a' \times h'$ = twice the area of the $\triangle A'B'C'$,

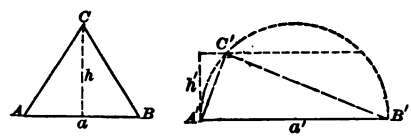
and $a \times h$ = twice the area of the $\triangle ABC$. § 403

$\therefore \triangle A'B'C' = \triangle ABC$. Q. E. F.

Ex. 386. *To transform a given triangle into an equivalent right triangle, having the hypotenuse equal to a given length.*

Let ABC be the given $\triangle$, a its base, and a' the given length.

To transform the $\triangle ABC$ into an equivalent rt. $\triangle$, having its hypotenuse equal to a'.



Construction. Draw h, the altitude of $\triangle ABC$.

Find h', the fourth proportional to a', a, and h. § 386

Construct a rt. $\triangle A'B'C'$, with the hypotenuse a', and the altitude upon the hypotenuse equal to h'. Ex. 186

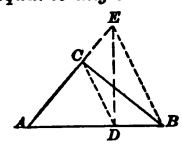
Then the $\triangle A'B'C'$ is the $\triangle$ required.

Proof. The proof is the same as in Ex. 385. Q. E. F.

Ex. 387. *To transform a triangle ABC into an equivalent triangle, having a side equal to a given length l, and an angle equal to angle BAC.*

Let ABC be the given $\triangle$, and l the given length.

To transform the $\triangle ABC$ into an equivalent $\triangle$ having one side equal to l, and one angle equal to $\angle BAC$.



Construction. Upon AB take AD equal to l, and draw CD.

Draw $BE \parallel$ to DC, meeting AC produced at E, and draw DE.

Then the $\triangle ADE$ is the $\triangle$ required.

Proof. In the $\triangle ABC$ and ADE, the $\triangle ADC$ is common.

$\triangle DBC = \triangle DCE$. § 404

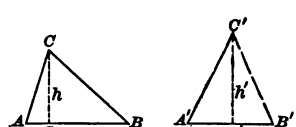
$\therefore \triangle ABC = \triangle ADE$. Ax. 2

Q. E. F.

Ex. 388. *To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.*

Let ABC be the given $\triangle$, a its base, and a' the given length.

To transform $\triangle ABC$ into an equivalent isosceles $\triangle$ having a base equal to a'.



Construction. Draw h, the altitude of the $\triangle ABC$.

Find h', the fourth proportional to a', a, and h. § 386

Take $A'B'$ equal to a'. At the middle point of $A'B'$ erect a $\perp$ equal to h', and draw $A'C'$ and $B'C'$.

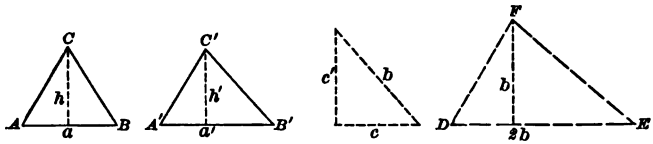
Then the $\triangle A'B'C'$ is the $\triangle$ required.

Proof. $A'C' = B'C'$. § 160

$\therefore$ the $\triangle A'B'C'$ is isosceles.

The remainder of the proof is the same as in Ex. 385. Q. E. F.

Ex. 389. *To construct a triangle equivalent to the sum of two given triangles.*



Let ABC and $A'B'C'$ be the given $\triangle$, and a and a' their bases.

To construct a $\triangle$ equivalent to the sum of the $\triangle$ ABC and $A'B'C'$.

Construction. Draw h and h', the altitudes of the $\triangle$ ABC and $A'B'C'$.

Find c, the mean proportional between a and $\frac{1}{2}h$, and c', the mean proportional between a' and $\frac{1}{2}h'$. § 388

Find b so that $b^2 = c^2 + c'^2$. § 417

Construct the $\triangle$ DEF with base equal to $2b$, and altitude equal to b.

Then the $\triangle$ DEF is the $\triangle$ required.

Proof. The area of the $\triangle DEF = \frac{1}{2}(2b \times b)$ § 403

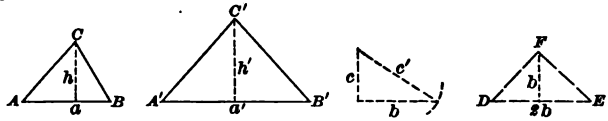
$= b^2 = c^2 + c'^2 = \frac{1}{2}(a \times h) + \frac{1}{2}(a' \times h')$. Const.

But the area of the $\triangle ABC = \frac{1}{2}(a \times h)$,

and the area of the $\triangle A'B'C' = \frac{1}{2}(a' \times h')$. § 403

$\therefore \triangle DEF = \triangle ABC + \triangle A'B'C'$. Q. E. F.

Ex. 390. *To construct a triangle equivalent to the difference of two given triangles.*



Let ABC and $A'B'C'$ be the given $\triangle$, and a and a' their bases.

To construct a $\triangle$ equivalent to the difference of the $\triangle$ $A'B'C'$ and ABC.

Construction. Draw h and h', the altitudes of the $\triangle$ ABC and $A'B'C'$.

Find c, the mean proportional between a and $\frac{1}{2}h$, and c', the mean proportional between a' and $\frac{1}{2}h'$. § 388

Find b so that $b^2 = c'^2 - c^2$. § 418

Construct the $\triangle$ DEF with base equal to $2b$, and altitude equal to b.

Then the $\triangle$ DEF is the $\triangle$ required.

Proof. The area of the $\triangle DEF = \frac{1}{2}(2b \times b)$ § 403

$= b^2 = c'^2 - c^2 = \frac{1}{2}(a' \times h') - \frac{1}{2}(a \times h)$. Const.

But the area of the $\triangle A'B'C' = \frac{1}{2}(a' \times h')$,

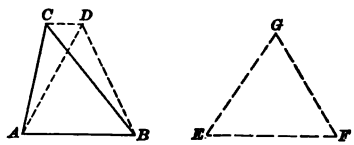
and the area of the $\triangle ABC = \frac{1}{2}(a \times h)$. § 403

$\therefore$ the $\triangle DEF = \triangle A'B'C' - \triangle ABC$. Q. E. F.

Ex. 391. *To transform a given triangle into an equivalent equilateral triangle.*

Let ABC be the given $\triangle$.

To transform the $\triangle ABC$ into an equivalent equilateral $\triangle$.



Construction. Draw AD, making the $\angle BAD$ equal to $\frac{1}{3}$ of a rt. $\angle$ (Ex. 137), meeting CD, drawn $\parallel$ to AB, at D, and draw BD.

Find EF, the mean proportional between AD and AB. § 388

On EF construct the equilateral $\triangle EFG$. § 312

Then the $\triangle EFG$ is the $\triangle$ required.

Proof. In the $\triangle ABD$ and EFG,

$\angle BAD = \angle E$. § 136

$\therefore \triangle ABD : \triangle EFG = AD \times AB : EG \times EF$. § 410

But $EG \times EF = \overline{EF}^2 = AD \times AB$. Const.

$\therefore \triangle ABD \backsim \triangle EFG$. Ax. 1

Also $\triangle ABD \backsim \triangle ABC$. § 404

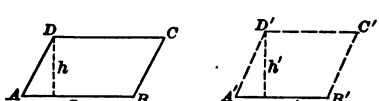
$\therefore \triangle EFG \backsim \triangle ABC$. Ax. 1

Q. E. F.

Ex. 392. *To transform a parallelogram into an equivalent parallelogram having one side equal to a given length.*

Let $ABCD$ be the given $\square$, a its base, and a' the given length.

To transform the $\square ABCD$ into an equivalent $\square$ having one side equal to a'.



Construction. Draw h, the altitude of the $\square ABCD$.

Find h', the fourth proportional to a', a, and h. § 386

Construct the $\square A'B'C'D'$ on a' as base, with h' as altitude.

Then the $\square A'B'C'D'$ is the $\square$ required.

Proof. $a' : a = h : h'$. Const.

$\therefore a' \times h' = a \times h$. § 327

But the area of the $\square A'B'C'D' = a' \times h'$,

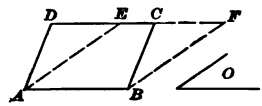
and the area of the $\square ABCD = a \times h$. § 400

$\therefore \square A'B'C'D' \backsim \square ABCD$. Ax. 1

Q. E. F.

Ex. 393. *To transform a parallelogram into an equivalent parallelogram having one angle equal to a given angle.*

Let $ABCD$ be the given $\square$, and O the given $\angle$.



To transform the $\square\ ABCD$ into an equivalent $\square$ having an $\angle$ equal to $\angle\ O$.

Construction. Construct the $\angle\ BAE$ equal to $\angle\ O$.

Draw $BF \parallel$ to AE, meeting DC produced at F.

Then the $\square\ ABFE$ is the $\square$ required.

Proof. $\square\ ABFE \fallingdotseq \square\ ABCD$. § 401

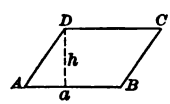
Q. E. F.

Ex. 394. *To transform a parallelogram into an equivalent rectangle having a given altitude.*

Let $ABCD$ be the given $\square$, a its base, and h' the given altitude.

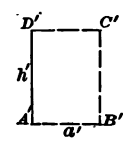
To transform the $\square\ ABCD$ into an equivalent rectangle with altitude h'.

Construction. Draw h, the altitude of the $\square\ ABCD$.



Find a', the fourth proportional to h', h, and a. § 386

Construct the rectangle $A'B'C'D'$, with base a' and altitude h'.



Then the rectangle $A'B'C'D'$ is the rectangle required.

Proof. $h' : h = a : a'$. Const.

$\therefore a' \times h' = a \times h$. § 327

But $a' \times h'$ = the area of the rectangle $A'B'C'D'$, § 398

and $a \times h$ = the area of the $\square\ ABCD$. § 400

$\therefore$ the rectangle $A'B'C'D' \fallingdotseq \square\ ABCD$. Ax. 1

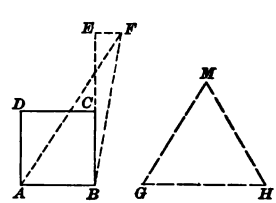
Q. E. F.

Ex. 395. *To transform a square into an equivalent equilateral triangle.*

Let $ABCD$ be the given square.

To transform the square $ABCD$ into an equivalent equilateral $\triangle$.

Construction. Prolong BC to E, making CE equal to BC.



Draw $EF \parallel$ to AB.

Draw AF, making the $\angle\ BAF$ equal to two thirds of a rt. $\angle$. Ex. 137

Draw BF.

Construct the equilateral $\triangle\ GHM$ equivalent to $\triangle\ ABF$. Ex. 391

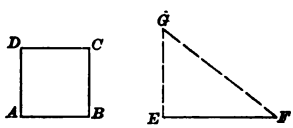
Then the $\triangle\ GHM$ is the $\triangle$ required.

Proof. The area of the $\triangle ABF = \frac{1}{2}(AB \times BE)$ § 403
$= \frac{1}{2}(AB \times 2\,BC) = \overline{AB}^2$. Const.
$\therefore \triangle ABF \simeq$ the square $ABCD$.
$\therefore \triangle GHM \simeq$ the square $ABCD$. Q. E. F.

Ex. 396. *To transform a square into an equivalent right triangle having one leg equal to a given length.*

Let $ABCD$ be the given square, and EF the given length.

To transform the square $ABCD$ into an equivalent rt. $\triangle$ having one leg equal to EF.



Construction. Find EG, the fourth proportional to EF, $2\,AB$, and AB. § 386

Construct the rt. $\triangle EFG$ with legs equal to EF and EG.

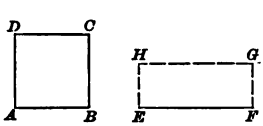
Then the $\triangle EFG$ is the $\triangle$ required.

Proof. $EF : 2\,AB = AB : EG$. Const.
$\therefore EF \times EG = 2\,\overline{AB}^2$. § 327
$\therefore \frac{1}{2}(EF \times EG) = \overline{AB}^2$. Ax. 7
But $\frac{1}{2}(EF \times EG)$ = the area of the $\triangle EFG$, § 403
and $\overline{AB}^2$ = the area of the square $ABCD$. § 398
$\therefore \triangle EFG \simeq$ the square $ABCD$. Q. E. F.

Ex. 397. *To transform a square into an equivalent rectangle having one side equal to a given length.*

Let $ABCD$ be the given square, and EF the given length.

To transform the square $ABCD$ into an equivalent rectangle having one side equal to EF.



Construction. Find EH, the third proportional to EF and AB. § 387

Construct the rectangle $EFGH$ with EF and EH as sides.

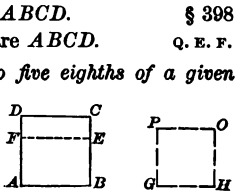
Then the rectangle $EFGH$ is the rectangle required.

Proof. $EF : AB = AB : EH$. Const.
$\therefore EF \times EH = \overline{AB}^2$. § 327
But $EF \times EH$ = the area of the rectangle $EFGH$, § 398
and $\overline{AB}^2$ = the area of the square $ABCD$. § 398
$\therefore$ the rectangle $EFGH \simeq$ the square $ABCD$. Q. E. F.

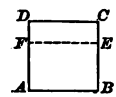
Ex. 398. *To construct a square equivalent to five eighths of a given square.*

Let $ABCD$ be the given square.

To construct a square equivalent to five eighths of the square $ABCD$.



Construction. Divide the side BC into eight equal parts. § 307



Take BE equal to five of these parts.

Draw $EF \parallel$ to BA.

Find GH, the mean proportional between AB and BE. § 388

On GH construct the square $GHOP$.

Then the square $GHOP$ is the square required.

Proof. $AB : GH = GH : BE$. Const.

$\therefore AB \times BE = \overline{GH}^2$. § 327

$\therefore$ the rectangle $ABEF \simeq$ the square $GHOP$. § 398

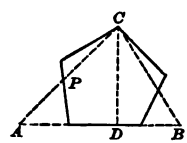
But the rectangle $ABEF \simeq \frac{5}{8}$ of the square $ABCD$. § 396

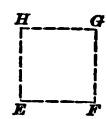
$\therefore$ the square $GHOP \simeq \frac{5}{8}$ of the square $ABCD$. Q. E. F.

Ex. 399. *To construct a square equivalent to three fifths of a given pentagon.*

Let P represent the given pentagon.

To construct a square equivalent to three fifths of P.





Construction. Construct the $\triangle ABC$ equivalent to P. § 420

Draw CD, the altitude of the $\triangle ABC$.

Divide AB into 5 equal parts. § 307

Find EF, the mean proportional between $\frac{3}{5} AB$ and $\frac{1}{2} CD$. § 388

On EF construct the square $EFGH$.

Then the square $EFGH$ is the square required.

Proof. $\frac{3}{5} AB : EF = EF : \frac{1}{2} CD$. Const.

$\therefore \frac{3}{5} (\frac{1}{2} AB \times CD) = \overline{EF}^2$. § 327

But $\frac{1}{2} AB \times CD =$ the area of the $\triangle ABC$, § 403

and $\overline{EF}^2 =$ the area of the square $EFGH$. § 398

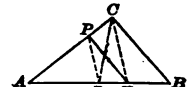
$\therefore$ the square $EFGH \simeq \frac{3}{5} \triangle ABC$.

$\therefore$ the square $EFGH \simeq \frac{3}{5} P$. Q. E. F.

Ex. 400. *To divide a given triangle into two equivalent parts by a line through a given point P in one of the sides.*

Let ABC be the given $\triangle$, and P the given point.

To draw a line through P so as to divide the $\triangle ABC$ into two equivalent parts.



Construction. Draw CD to the middle point of AB, and draw PD.

Draw $CE \parallel$ to PD, and draw PE.

Then PE is the line required.

Proof. $\triangle ADC \backsim \triangle BDC$. § 404

$\therefore \triangle ADC \backsim \frac{1}{2} \triangle ABC$. § 405

In the $\triangle AEP$ and ADC, the $\triangle ADP$ is common,

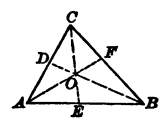
and $\triangle DEP \backsim \triangle DPC$. § 404

$\therefore \triangle AEP \backsim \triangle ADC \backsim \frac{1}{2} \triangle ABC$. Q. E. F.

Ex. 401. *To find a point within a triangle such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.*

Let ABC be the given $\triangle$.

To find a point within the $\triangle ABC$ such that the lines joining this point to the vertices shall divide the $\triangle$ into three equivalent parts.



Construction. Draw the medians AF, BD, and CE. These medians meet at a point O. Ex. 27

Then O is the point required.

Proof. $\triangle AEC \backsim \frac{1}{2} \triangle ABC$, and $\triangle ABD \backsim \frac{1}{2} \triangle ABC$. § 405

$\therefore \triangle ABD \backsim \triangle AEC$.

Subtract from each $\triangle$ the common quadrilateral $AEOD$.

Then $\triangle EBO \backsim \triangle DOC$. Ax. 3

$\triangle AEO \backsim \triangle EBO$, and $\triangle AOD \backsim \triangle DOC$. § 404

$\therefore \triangle AEO \backsim \triangle AOD$. Ax. 1

$\therefore \triangle AEO + \triangle EBO \backsim \triangle AOD + \triangle DOC$. Ax. 2

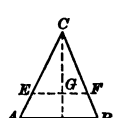
Or $\triangle ABO \backsim \triangle AOC$.

In like manner, $\triangle ABO \backsim \triangle BOC$. Q. E. F.

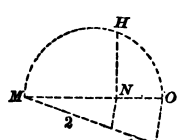
Ex. 402. *To divide a given triangle into two equivalent parts by a line parallel to one of the sides.*

Let ABC be the given $\triangle$.

To divide the $\triangle ABC$ into two equivalent parts by a line $\parallel$ to AB.



Construction. Draw CD, the altitude of the $\triangle ABC$.



Take MN equal to CD.

Find NH, so that $\overline{MN}^2 : \overline{NH}^2 = 2 : 1$. § 427

Take CG equal to NH, and draw EF through $G \parallel$ to AB.

Then EF is the line required.

Proof. Since EF is $\parallel$ to AB, the $\triangle ABC$ and EFC are mutually equiangular and similar. § 354

$\therefore \triangle ABC : \triangle EFC = \overline{CD}^2 : \overline{CG}^2$ § 413

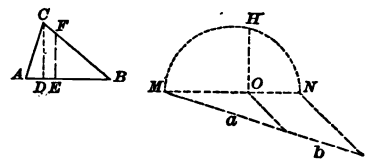
$= \overline{MN}^2 : \overline{NH}^2 = 2 : 1$. Const.

Q. E. F.

Ex. 403. *To divide a given triangle into two equivalent parts by a line perpendicular to one of the sides.*

Let ABC be the given $\triangle$.

To divide $\triangle ABC$ into two equivalent parts by a line $\perp$ to AB.



Construction. Draw CD, the altitude of $\triangle ABC$.

Take a equal to $2\,DB$, b equal to AB, MO equal to BC.

Find OH, so that

$$\overline{MO}^2 : \overline{OH}^2 = a : b. \quad \text{§ 427}$$

Cut off BF equal to OH, and draw $FE \perp$ to AB.

Then FE is the line required.

Proof. Now FE is $\parallel$ to CD. § 104

$\therefore$ $\triangle DBC$ and EBF are similar. § 354

$$\therefore \frac{\triangle DBC}{\triangle EBF} = \frac{\overline{BC}^2}{\overline{BF}^2} \quad \text{§ 411}$$

$$= \frac{\overline{MO}^2}{\overline{OH}^2} = \frac{a}{b} = \frac{2\,DB}{AB}. \quad \text{Const.}$$

That is, $$\frac{\triangle DBC}{\triangle EBF} = \frac{2\,DB}{AB}.$$

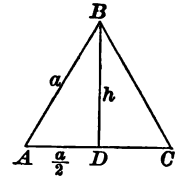
But $$\frac{\triangle ABC}{\triangle DBC} = \frac{AB}{DB}. \quad \text{§ 405}$$

Multiply the corresponding members of these equations.

Then $$\frac{\triangle ABC}{\triangle EBF} = \frac{2}{1}. \quad \text{Q. E. F.}$$

Ex. 404. *To find the area of an equilateral triangle in terms of its side.*

Denote the side by a, the altitude by h, and the area by S.



Then $$h = \frac{a}{2}\sqrt{3}. \quad \text{Ex. 328}$$

But $$S = \tfrac{1}{2}(a \times h). \quad \text{§ 403}$$

$$\therefore S = \frac{a}{2} \times \frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}.$$

Ex. 405. *To find the area of a triangle in terms of its sides.*

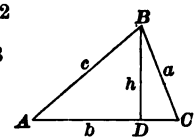
Denote the sides by a, b, c, the altitude by h, and the area by S.

Now $h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. Ex. 312

But $S = \frac{1}{2}(b \times h)$. § 403

$\therefore S = \frac{b}{2} \times \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{s(s-a)(s-b)(s-c)}$.



Ex. 406. *To find the area of a triangle in terms of the radius of the circumscribed circle.*

If R denotes the radius of the circumscribed circle, and h the altitude of the triangle, we have, by § 384,

$b \times c = 2R \times h$.

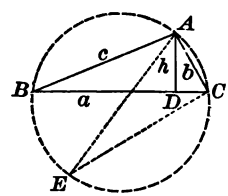
Multiply by a and we have,

$a \times b \times c = 2R \times a \times h$.

But $a \times h = 2S$. § 403

$\therefore a \times b \times c = 4R \times S$.

$\therefore S = \frac{abc}{4R}$.



Ex. 407. *Find the area of a right triangle if the length of the hypotenuse is* 17 *feet, and the length of one leg is* 8 *feet.*

In the rt. $\triangle ABC$,

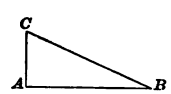
$AC = 8$ feet, and $BC = 17$ feet.

$AB = \sqrt{\overline{BC}^2 - \overline{AC}^2} = \sqrt{17^2 - 8^2} = 15$. § 372

The area of the $\triangle ABC = \frac{1}{2}(AB \times AC)$

$= \frac{1}{2}(15 \times 8) = 60$. § 403

60 sq. ft. *Ans.*



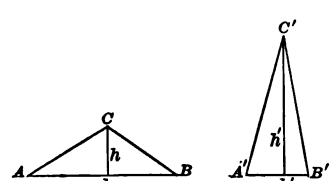
Ex. 408. *Find the ratio of the altitudes of two equivalent triangles, if the base of one is three times that of the other.*

Let the equivalent $\triangle ABC$ and $A'B'C'$ have the bases b and b', and the altitudes h and h', respectively.

Let b equal $3b'$.

$\triangle ABC : \triangle A'B'C' = b \times h : b' \times h'$.

§ 405



But $\triangle ABC \simeq \triangle A'B'C'$. Hyp.

$\therefore b \times h = b' \times h'$. § 405

$\therefore b' : b = h : h'$. § 329

But $b' : b = 1 : 3$. Hyp.

$\therefore h : h' = 1 : 3$. Ax. 1

Ex. 409. *The bases of a trapezoid are* 8 *feet and* 10 *feet, and the altitude is* 6 *feet. Find the base of an equivalent rectangle that has an equal altitude.*

Let x = the length of the base of the rectangle.

The area of the trapezoid $= 6 \times \frac{1}{2}(8 + 10)$. § 407

The area of the rectangle $= 6x$. § 398

$\therefore 6x = 6 \times \frac{1}{2}(8 + 10)$.

$6x = 54$.

$x = 9$. 9 feet. *Ans.*

Ex. 410. *Find the area of a rhombus, if the sum of its diagonals is* 12 *feet, and their ratio is* 3 : 5.

Let $5x$ = the length of one diagonal.

Then $3x$ = the length of the other.

$\therefore 5x + 3x = 12$.

$\therefore x = 1\frac{1}{2}$.

$\therefore 5x = 7\frac{1}{2}$,

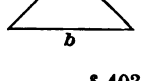
$3x = 4\frac{1}{2}$.

Therefore, the area $= \frac{1}{2}(7\frac{1}{2} \times 4\frac{1}{2}) = 16\frac{7}{8}$. Ex. 359

$16\frac{7}{8}$ square feet. *Ans.*

Ex. 411. *Find the area of an isosceles right triangle, if the hypotenuse is* 20 *feet.*

In the isosceles rt. $\triangle$, let the hypotenuse b equal 20 feet.



$2a^2 = b^2$. § 371

$\therefore a = b\sqrt{\frac{1}{2}} = 20\sqrt{\frac{1}{2}}$.

The area of the $\triangle = \frac{1}{2}a^2 = \frac{1}{2}(20\sqrt{\frac{1}{2}})^2 = 100$. § 403

100 square feet. *Ans.*

Ex. 412. *In a right triangle, the hypotenuse is* 13 *feet, one leg is* 5 *feet. Find the area.*

Let x = the length of the other leg.

$\therefore x = \sqrt{13^2 - 5^2} = 12$. § 372

The area $= \frac{1}{2}(5 \times 12) = 30$. § 403

30 square feet. *Ans.*

Ex. 413. *Find the area of an isosceles triangle, if base* $= b$, *and leg* $= c$.

Let h be the altitude.

Then $h = \sqrt{c^2 - (\frac{1}{2}b)^2}$ § 372

$= \sqrt{\frac{4c^2 - b^2}{4}} = \frac{1}{2}\sqrt{(2c + b)(2c - b)}$.

The area $= \frac{1}{2}(h \times b) = \frac{1}{2}b\sqrt{(2c + b)(2c - b)}$. § 403

Ex. 414. *Find the area of an equilateral triangle, if one side = 8 feet.*

$$\text{The area} = \frac{a^2\sqrt{3}}{4} = \frac{8^2\sqrt{3}}{4} = 16\sqrt{3} = 27.712. \qquad \text{Ex. 404}$$

27.712 square feet. *Ans.*

Ex. 415. *Find the area of an equilateral triangle, if the altitude = h.*

$$a = \frac{2h}{3}\sqrt{3}. \qquad \text{Ex. 329}$$

$$\text{The area} = \tfrac{1}{2}(h \times a) = \tfrac{1}{2}(h \times \frac{2h}{3}\sqrt{3}) = \frac{h^2}{3}\sqrt{3}. \qquad \text{§ 403}$$

Ex. 416. *A house is 40 feet long, 30 feet wide, 25 feet high to the roof, and 35 feet high to the ridge-pole. Find the number of square feet in its entire exterior surface.*

The area of the four sides $= 2 \times (40 + 30) \times 25 = 3500.$ § 398

The area of the gable ends $= 2 \times \tfrac{1}{2}(10 \times 30) = 300.$ § 403

The length of the rafters $= \sqrt{10^2 + 15^2} = 18.027.$ § 371

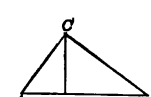
The area of the roof $= 2 \times 18.027 \times 40 = 1442.61.$

The entire area $= (3500 + 300 + 1442.16)$ square feet

$= 5242.16$ square feet.

Ex. 417. *The sides of a right triangle are as 3 : 4 : 5. The altitude upon the hypotenuse is 12 feet. Find the area.*

In the rt. $\triangle ABC$, let DC, the altitude upon the hypotenuse, equal 12 feet, and let



$AC : BC : AB$ be as $3 : 4 : 5$.

Let AB equal $5x$.

Then $AC = 3x$, $BC = 4x$.

$\therefore 5x : 3x = 3x : AD.$ § 367

$\therefore AD = \frac{9x}{5}.$ § 327

$\therefore DB = 5x - \frac{9x}{5} = \frac{16x}{5}.$

$\frac{9x}{5} : 12 = 12 : \frac{16x}{5}.$ § 367

$\therefore \frac{144x^2}{25} = 144.$ § 327

$x^2 = 25.$

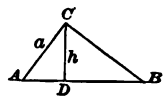
$x = 5.$

$\therefore AB = 25.$

$\text{The area} = \tfrac{1}{2}(12 \times 25) = 150.$ § 403

150 square feet. *Ans.*

Ex. 418. *Find the area of a right triangle, if one leg = a, and the altitude upon the hypotenuse = h.*



$AD = \sqrt{a^2 - h^2}$. § 372

$AB : a = a : AD$

$= a : \sqrt{a^2 - h^2}$. § 367

$\therefore AB = \dfrac{a^2}{\sqrt{a^2 - h^2}}$. § 327

The area of the $\triangle ABC = \frac{1}{2}(h \times AB)$ § 403

$= \frac{1}{2}\left(h \times \dfrac{a^2}{\sqrt{a^2 - h^2}}\right) = \dfrac{a^2 b}{2(a^2 - b^2)}\sqrt{a^2 - h^2}$.

Ex. 419. *Find the area of a triangle, if the lengths of the sides are* 104 *feet,* 111 *feet, and* 175 *feet.*

The area $= \sqrt{s(s-a)(s-b)(s-c)}$ Ex. 405

$= \sqrt{195(195-104)(195-111)(195-175)}$

$= 5460$. 5460 square feet. *Ans.*

Ex. 420. *The area of a trapezoid is* 700 *square feet. The bases are* 30 *feet and* 40 *feet, respectively. Find the altitude.*

Let h be the altitude.

The area of the trapezoid $= \frac{1}{2}(40 + 30)\,h = 35h$. § 407

$\therefore 700 = 35h$.

$\therefore h = 20$. 20 feet. *Ans.*

Ex. 421. *ABCD is a trapezium; AB* = 87 *feet, BC* = 119 *feet, CD* = 41 *feet, DA* = 169 *feet, AC* = 200 *feet. Find the area.*

The area of the $\triangle ABC$ in square feet

$= \sqrt{203(203-87)(203-119)(203-200)}$ Ex. 405

$= 2436$.

The area of the $\triangle ABC$ in square feet

$= \sqrt{205(205-200)(205-41)(205-169)}$

$= 2460$.

$\therefore$ the area of $ABCD = (2436 + 2460)$ square feet $= 4896$ square feet.

Ex. 422. *What is the area of a quadrilateral circumscribed about a circle whose radius is* 25 *feet, if the perimeter of the quadrilateral is* 400 *feet? What is the area of a hexagon that has a perimeter of* 400 *feet and is circumscribed about the same circle of* 25 *feet radius?*

The area of the quadrilateral

$= \frac{1}{2}(400 \times 25)$ square feet $= 5000$ square feet. Ex. 361

The hexagon has the same area. Ex. 361

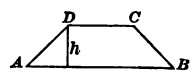
Ex. 423. *The base of a triangle is* 15 *feet, and its altitude is* 8 *feet. Find the perimeter of an equivalent rhombus, if the altitude is* 6 *feet.*

The area of the $\triangle = \frac{1}{2}(8 \times 15)$ square feet § 403
$= 60$ square feet.
One side of the rhombus $= (60 \div 6)$ feet $= 10$ feet.
The perimeter $= 4 \times 10$ feet $= 40$ feet.

Ex. 424. *Upon the diagonal of a rectangle* 24 *feet by* 10 *feet a triangle equivalent to the rectangle is constructed. What is its altitude?*

The area of the rectangle $= (24 \times 10)$ sq. ft. $= 240$ sq. ft. § 398
The diagonal of the rectangle $= \sqrt{24^2 + 10^2}$ ft. $= 26$ ft. § 371
Let h denote the altitude of the $\triangle$.
Then the area of the $\triangle = \frac{1}{2}(h \times 26)$. § 403
$\therefore 240 = \frac{1}{2}(h \times 26) = 13h.$
$\therefore h = 18\frac{6}{13}$. $18\frac{6}{13}$ feet. *Ans.*

Ex. 425. *Find the side of a square equivalent to a trapezoid whose bases are* 56 *feet and* 44 *feet, and each leg is* 10 *feet.*



In the trapezoid $ABCD$, $AB = 56$, $DC = 44$, and $AD = BC = 10$.

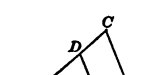
Let h denote the altitude of the trapezoid $ABCD$, and x a side of the required square.

Since the trapezoid $ABCD$ is isosceles,

$$h = \sqrt{\overline{AD}^2 - (\tfrac{1}{2}AB - \tfrac{1}{2}DC)^2} \quad \text{§ 372}$$
$$= \sqrt{10^2 - 6^2} = 8.$$

The area of the trapezoid $ABCD$ in sq. ft. $= \frac{1}{2}(56 + 44)8 = 400$. § 407
$\therefore x^2 = 400.$
$\therefore x = 20.$ 20 feet. *Ans.*

Ex. 426. *Through a point P in the side AB of a triangle ABC, a line is drawn parallel to BC so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.*

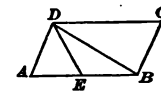


In the $\triangle ABC$, let PD be drawn $\parallel$ to BC, dividing the $\triangle ABC$ into two equivalent parts.

$$\triangle APD : \triangle ABC = \overline{AP}^2 : \overline{AB}^2 = 1 : 2. \quad \text{§ 411}$$
$$\therefore \overline{AP}^2 = \tfrac{1}{2}\overline{AB}^2.$$
$$\therefore AP = AB\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}AB\sqrt{2}.$$

Ex. 427. *What part of a parallelogram is the triangle cut off by a line from one vertex to the middle point of one of the opposite sides?*

In the $\square ABCD$, let DE be drawn to the middle point of AB.



$\triangle ADE = \frac{1}{2} \triangle ADB$. § 405

But $\triangle ADB = \frac{1}{2} \square ABCD$. § 179

$\therefore \triangle ADE = \frac{1}{4} \square ABCD$.

Ex. 428. *In two similar polygons, two homologous sides are* 15 *feet and* 25 *feet. The area of the first polygon is* 450 *square feet. Find the area of the second polygon.*

Let S denote the area of the first polygon, and S' the area of the second.

Then $S : S' = 15^2 : 25^2$. § 412

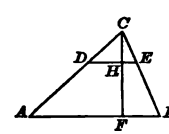
$450 : S' = 15^2 : 25^2$.

$\therefore S' = 1250$. 1250 square feet. *Ans.*

Ex. 429. *The base of a triangle is* 32 *feet, its altitude* 20 *feet. What is the area of the triangle cut off by a line parallel to the base at a distance of* 15 *feet from the base?*

In the $\triangle ABC$, $AB = 32$, $CF = 20$.

Let DE be $\parallel$ to AB, cutting off HF equal to 15.



The area of the $\triangle ABC = \frac{1}{2}(32 \times 20)$ square feet

$= 320$ square feet. § 403

$CH = 20$ feet $- 15$ feet $= 5$ feet.

$\therefore \triangle DEC : \triangle ABC = \overline{CH}^2 : \overline{CF}^2$ § 413

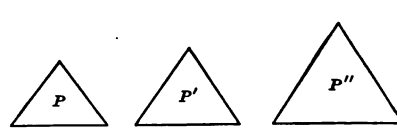
$= 5^2 : 20^2 = 1 : 16$.

$\therefore$ the area of $\triangle DEC = \frac{1}{16}$ of 320 square feet $= 20$ square feet.

Ex. 430. *The sides of two equilateral triangles are* 3 *feet and* 4 *feet. Find the side of an equilateral triangle equivalent to their sum.*

Let P and P' be two equilateral $\triangle$ whose sides are 3 feet and 4 feet, respectively.

Let P'' be the required equilateral $\triangle$, and let x denote one side of P''.



Then $\frac{P}{P''} = \frac{3^2}{x^2}$,

and $\frac{P'}{P''} = \frac{4^2}{x^2}$. § 411

$\therefore \frac{P + P'}{P''} = \frac{3^2 + 4^2}{x^2}$. Ax. 2

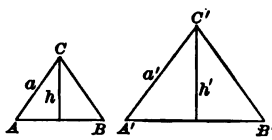
But $P + P' = P''$. Hyp.

$\therefore 3^2 + 4^2 = x^2$.

$\therefore x = 5$. 5 feet. *Ans.*

Ex. 431. *If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?*

In the equilateral $\triangle ABC$ and $A'B'C'$, let a, a side of ABC, equal h', the altitude of $A'B'C'$.



$\triangle ABC : \triangle A'B'C' = h^2 : h'^2$. § 413

But $h^2 = \frac{3a^2}{4}$, Ex. 404

and $h'^2 = a^2$. Hyp.

Substituting, $\triangle ABC : \triangle A'B'C' = \frac{3a^2}{4} : a^2 = 3 : 4$.

Ex. 432. *The sides of a triangle are* 10 *feet*, 17 *feet, and* 21 *feet. Find the areas of the parts into which the triangle is divided by the bisector of the angle formed by the first two sides.*

In the $\triangle ABC$, $AC = 10$ feet, $BC = 17$ feet, and $AB = 21$ feet.

Let CD bisect the $\angle C$.

The area of the $\triangle ABC$

$= \sqrt{24(24-10)(24-17)(24-21)}$ square feet Ex. 405

$= 84$ square feet.

$\triangle ADC : \triangle BDC = 10 \times CD : 17 \times CD$

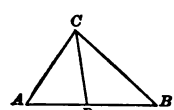
$= 10 : 17$. § 410

$\therefore \triangle ADC = \frac{10}{27}$ of $\triangle ABC = \frac{10}{27}$ of 84 square feet

$= 31\frac{1}{9}$ square feet,

and $\triangle BDC = \frac{17}{27}$ of $\triangle ABC = \frac{17}{27}$ of 84 square feet

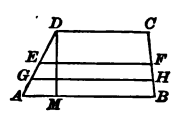
$= 52\frac{8}{9}$ square feet.



Ex. 433. *In a trapezoid one base is* 10 *feet, the altitude is* 4 *feet, the area is* 32 *square feet. Find the length of a line drawn between the legs parallel to the bases and distant* 1 *foot from the lower base.*

In the trapezoid $ABCD$, $AB = 10$ feet, $DM = 4$ feet, and the area $= 32$ square feet.

Let EF be the median and GH be a line parallel to AB and distant 1 foot from AB.



The area of the trapezoid $ABCD$

$= EF \times DM$. § 408

$\therefore 32 = EF \times 4$.

$\therefore EF = 8$.

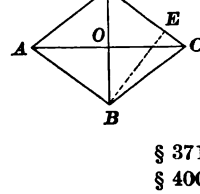
But EF bisects DM. § 187

Therefore, in the trapezoid $ABFE$,

GH is the median and $= \frac{1}{2}(10 + 8)$ feet $= 9$ feet. § 190

Ex. 434. *The diagonals of a rhombus are* 90 *yards and* 120 *yards, respectively. Find the area, the length of one side, and the perpendicular distance between two parallel sides.*

In the rhombus $ABCD$, $BD = 90$ yards, and $AC = 120$ yards.



Area of rhombus $ABCD$

$= \frac{1}{2} AC \times BD$ Ex. 359

$= \frac{1}{2}(120 \times 90)$ square yards

$= 5400$ square yards.

AC and BD bisect each other at rt. $\measuredangle$. § 184, Ex. 71

$\therefore AO = 60$ yards and $BO = 45$ yards.

$\therefore AB = \sqrt{60^2 + 45^2}$ yards $= 75$ yards. § 371

The area of the rhombus $ABCD = AB \times BE$. § 400

Substituting, $5400 = 75 \times BE$.

$\therefore BE = (5400 \div 75)$ yards $= 72$ yards.

Ex. 435. *Find the number of square feet of carpet that are required to cover a triangular floor whose sides are, respectively,* 26 *feet,* 35 *feet, and* 51 *feet.*

The area of the $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ Ex. 405

$= \sqrt{56 \times 30 \times 21 \times 5}$ square feet $= 420$ square feet.

Ex. 436. *If the altitude h of a triangle is increased by a length m, how much must be taken from the base a that the area may remain the same?*

Let $x =$ the length taken from the base.

Then $h + m =$ the increased altitude,

and $a - x =$ the diminished base.

$\therefore \frac{1}{2}(h \times a) =$ the area of the first $\triangle$, § 403

and $\frac{1}{2}(h + m)(a - x) =$ the area of the second $\triangle$.

$\therefore \frac{1}{2}(h \times a) = \frac{1}{2}(h + m)(a - x)$.

$ha = ha - hx + am - mx$.

$hx + mx = am$.

$\therefore x = \dfrac{am}{h + m}$.

Ex. 437. *Find the area of a right triangle, having given the segments p, q, into which the hypotenuse is divided by a perpendicular drawn to the hypotenuse from the vertex of the right angle.*

Let h denote the altitude on the hypotenuse.

Then $p : h = h : q$. § 367

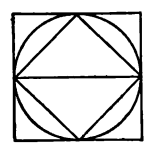
$\therefore h = \sqrt{p \times q}$. § 327

The area of the $\triangle = \frac{1}{2}(p + q)h = \frac{1}{2}(p + q)\sqrt{p \times q}$. § 403

Ex. 438. *The area of a circumscribed square is equal to twice the area of the inscribed square.*

Let S denote the side of the circumscribed square, s the side, and d the diagonal of the inscribed square.

To prove that $S^2 = 2s^2$.



Proof. $\frac{d}{s} = \sqrt{2}$. § 373

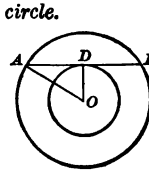
But d = diameter of circle $= S$.

$\therefore \frac{S}{s} = \sqrt{2}$, or $\frac{S^2}{s^2} = 2$.

$\therefore S^2 = 2s^2$. Q. E. D.

Ex. 439. *The area of a circular ring is equal to that of a circle whose diameter is a chord of the outer circle tangent to the inner circle.*

Let AB be a chord to the outer ⊙ tangent to the inner ⊙ at the point D. Let the radii of the outer and inner ⊙ be R and r, respectively.



To prove that area of ring $= \frac{1}{4}\pi \times \overline{AB}^2$.

Proof. Area of outer circle $= \pi R^2$. § 463

Area of inner circle $= \pi r^2$. § 463

$\therefore$ Area of ring $= \pi(R^2 - r^2)$.

But $\overline{OA}^2 - \overline{OD}^2 = \overline{AD}^2 = \frac{1}{4}\overline{AB}^2$. § 372

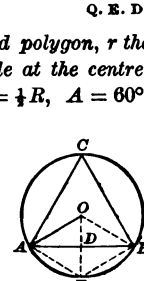
That is, $R^2 - r^2 = \frac{1}{4}\overline{AB}^2$.

$\therefore \pi(R^2 - r^2) = \frac{1}{4}\pi \times \overline{AB}^2$.

$\therefore$ area of ring $= \frac{1}{4}\pi \times \overline{AB}^2$. Q. E. D.

Ex. 440. *If R denotes the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre, show that in a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.*

Let ABC be an equilateral triangle inscribed in the circle whose centre is O.



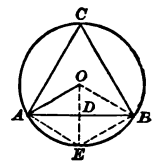
Let OD be perpendicular to AB.

Then $AO = R$, $OD = r$, $AB = a$, $\angle CAB = A$, and $\angle AOB = C$.

1. To prove that $a = R\sqrt{3}$.

Proof. Produce OD to meet the arc AB in E.

Draw AE, BO, BE.



E is the middle point of the arc AB. § 245

$\therefore AO = AE = BO = BE$. § 469

$\therefore AB$ and OE bisect each other at D. § 161

$\therefore \overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2$. § 372

That is, $(\frac{1}{2}a)^2 = R^2 - (\frac{1}{2}R)^2$.

$\therefore \frac{1}{4}a^2 = \frac{3}{4}R^2$.

$\therefore a = R\sqrt{3}$.

2. To prove that $r = \frac{1}{2}R$.

Proof. Since AB bisects OE, $OD = \frac{1}{2}OE$.

That is, $r = \frac{1}{2}R$.

3. To prove that $A = 60°$.

Proof. The $\triangle ABC$ is equiangular. § 146

$\therefore A = 60°$. § 146

4. To prove that $C = 120°$.

Proof. $C = \frac{1}{3}$ of $360° = 120°$. § 436

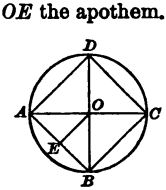
Q. E. D.

Ex. 441. *If R denotes the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre, show that in an inscribed square,*

$$a = R\sqrt{2},\ r = \tfrac{1}{2}R\sqrt{2},\ A = 90°,\ C = 90°.$$

Let $ABCD$ be an inscribed square, O the centre, and OE the apothem.

Then $AO = R$, $AB = a$, $OE = r$.



1. To prove that $a = R\sqrt{2}$.

Proof. $AO = \frac{1}{2}AB\sqrt{2}$. § 373

$\therefore AB = AO\sqrt{2}$.

That is, $a = R\sqrt{2}$.

2. To prove that $r = \frac{1}{2}R\sqrt{2}$.

Proof. $OE = \frac{1}{2}a = \frac{1}{2}R\sqrt{2}$.

That is, $r = \frac{1}{2}R\sqrt{2}$.

3. $A = 90°$. § 290

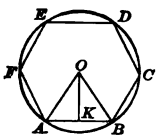
4. $C = \frac{1}{4}$ of $360° = 90°$. § 436

Q. E. D.

Ex. 442. *If R denotes the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre, show that in a regular inscribed hexagon*

$$a = R,\ r = \tfrac{1}{2}R\sqrt{3},\ A = 120°,\ C = 60°.$$

Let $ABCDEF$ be a regular hexagon inscribed in a $\odot$ whose centre is O. Let OK be the apothem.



Then $AO = R$, $AB = a$, $OK = r$.

1. $a = R$. § 469

2. $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2$. § 372

That is, $r^2 = R^2 - (\frac{1}{2}R)^2 = \frac{3R^2}{4}$.

$\therefore r = \frac{1}{2}R\sqrt{3}$.

3. $A = \angle FAB = 120°$, since it is measured by $\frac{1}{2}$ of $\frac{1}{3}$ of the circumference. § 289

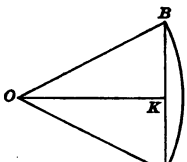
4. $C = \frac{1}{6}$ of $360° = 60°$. § 436

Q. E. D.

Ex. 443. *If R denotes the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre, show that in a regular inscribed decagon*

$$a = \frac{R(\sqrt{5}-1)}{2},\ r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}},\ A = 144°,\ C = 36°.$$

Let AB be one side of a regular decagon inscribed in a $\odot$ whose centre is O. Let OK be the apothem.



Then $OA = R$, $OK = r$, $AB = a$.

1. $OA : AB = AB : OA - AB$. § 472

That is, $R : a = a : R - a$.

$\therefore a^2 = R^2 - aR$. § 327

$4a^2 + 4aR + R^2 = 5R^2$.

$2a + R = R\sqrt{5}$.

$\therefore a = \frac{1}{2}R(\sqrt{5}-1)$.

2. $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2$. § 372

That is, $r^2 = R^2 - (\frac{1}{2}a)^2 = R^2 - \left\{\frac{R(\sqrt{5}-1)}{4}\right\}^2$.

$r^2 = R^2 - \frac{R^2(6-2\sqrt{5})}{16}$.

$r^2 = \frac{R^2(10+2\sqrt{5})}{16}$.

$\therefore r = \frac{1}{4}R\sqrt{10+2\sqrt{5}}$.

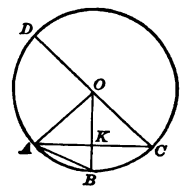
3. $A = 2\angle OAB = 144°$, since it is measured by $\frac{1}{2}$ of $\frac{8}{10}$ of the circumference. § 289

4. $C = \frac{1}{10}$ of $360° = 36°$. § 436

Q. E. D.

Ex. 444. *The area of an inscribed regular octagon is equal to that of a rectangle whose sides are equal to the sides of the inscribed and the circumscribed squares.*

Let S be the area and AB the side of an inscribed regular octagon, AC be the side of the inscribed square, and CD the diameter of the circle or the side of the circumscribed square.



To prove that $S = CD \times AC$.

Proof. Now $BO = \frac{1}{2} CD$,

and $AK = \frac{1}{2} AC$.

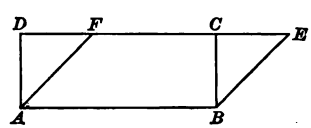
Area of $\triangle ABO = \frac{1}{2} BO \times AK = \frac{1}{8} CD \times AC$. § 403

But the $\triangle ABO$ is $\frac{1}{8}$ of the octagon.

$\therefore S = 8 \times \frac{1}{8} CD \times AC = CD \times AC$. Q. E. D.

Ex. 445. *Of all equivalent parallelograms that have equal bases, the rectangle has the minimum perimeter.*

Let $ABCD$ be a rectangle, and $ABEF$ any other $\square$ on the same base AB, with the same altitude.



To prove that

$2AB + 2BC < 2AB + 2BE$.

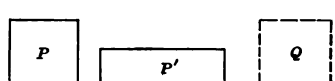
Proof.

$BC < BE$.	§ 97
$\therefore 2BC < 2BE$.	Ax. 6
$\therefore 2AB + 2BC < 2AB + 2BE$.	Ax. 4

Q. E. D.

Ex. 446. *Of all equivalent rectangles, the square has the minimum perimeter.*

Let P be a square, and P' any other rectangle equivalent to P.



To prove that the perimeter of P is less than the perimeter of P'.

Proof. Construct the square Q, having the same perimeter as P'.

Then	$Q > P'$.	§ 489
But	$P = P'$.	Hyp.
	$\therefore Q > P$.	

Since Q is greater than P, and they are both squares, the perimeter of Q must be greater than the perimeter of P.

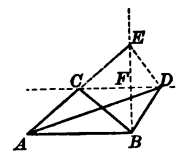
But the perimeter of Q equals the perimeter of P'. Const.

$\therefore$ the perimeter of P' is greater than the perimeter of P. Q. E. D.

Ex. 447. *Of all triangles that have the same base and the same altitude, the isosceles has the minimum perimeter.*

Let ABC be an isosceles $\triangle$, and ABD any other $\triangle$, having AB for a base, and its vertex D in a line $CD \parallel$ to AB.

To prove that $AB + AC + CB < AB + AD + DB$.



Proof. Draw $BE \perp$ to AB, meeting AC produced at E, and draw DE, meeting CD at F.

BE is $\perp$ to CD.	§ 107
$\angle ECF = \angle CAB$.	§ 112
$\angle CAB = \angle CBA$.	§ 145
$\angle CBA = \angle BCF$.	§ 110
$\therefore \angle ECF = \angle BCF$.	Ax. 1
$\therefore$ the rt. $\triangle ECF =$ rt. $\triangle BCF$.	§ 142
$\therefore FE = FB$.	§ 128
$\therefore CE = CB$ and $DE = DB$.	§ 160
But $AE < AD + DE$.	§ 138
That is, $AC + CB < AD + DB$.	
$\therefore AB + AC + CB < AB + AD + DB$.	Ax. 4

Q. E. D.

Ex. 448. *Of all triangles that can be inscribed in a given circle, the equilateral is the maximum and has the maximum perimeter.*

Let ABC be a triangle inscribed in the given circle.

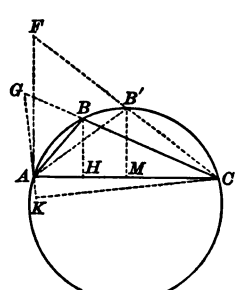
1. To prove that $\triangle ABC$ is the maximum when equilateral.

Proof. If $\triangle ABC$ is not equilateral, at least two sides are unequal.

Let BC be greater than BA.

Draw $BH \perp$ to AC.

Erect MB' the $\perp$ bisector of AC, meeting the circumference at B', and draw AB' and CB'.



Since MB' passes through the centre of the $\odot$, § 248

the tangent through B' is $\perp$ to MB', § 254

and therefore is $\parallel$ to AC. § 104

B' lies in this tangent and B lies between the tangent and AC. § 220

$\therefore B'M > BH$.

$\therefore \triangle AB'C > \triangle ABC$. § 405

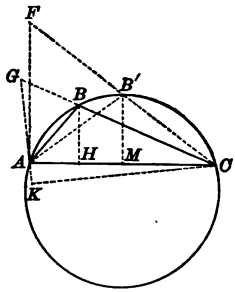
But $AB' = CB'$. § 160

Therefore, of all $\triangle$ inscribed in the $\odot$ having a base equal to AB, the maximum is the isosceles.

If now AB' and AC are unequal, considering $B'C$ the base of the $\triangle$, the maximum $\triangle$ is isosceles.

That is, the $\triangle ABC$ is the maximum when all three sides are equal; that is, when equilateral.

2. To prove that $\triangle ABC$ has the maximum perimeter when the $\triangle$ is equilateral.



Proof. Produce CB' to F, making $B'F$ equal to $B'A$, and produce CB to G, making BG equal to BA.

Draw AF and AG, and draw $CK \perp$ to GA produced.

Now FA is $\perp$ to AC. Ex. 43

$\angle F = \angle B'AF$, and $\angle G = \angle BAG$. § 145

$\therefore \angle AB'C = 2\angle F$, and $\angle ABC = 2\angle G$. § 137

Now $\angle AB'C = \angle ABC$. § 293

$\therefore \angle F = \angle G$. Ax. 7

$\therefore$ the rt. $\triangle FAC$ and GKC are similar. § 356

$\therefore CF : CG = CA : CK$. § 351

That is, $CB' + B'A : CB + BA = CA : CK$.

But $CA > CK$. § 97

Therefore, the perimeter of the $\triangle AB'C$ is greater than the perimeter of the $\triangle ABC$.

That is, the perimeter of a $\triangle$ inscribed in the $\odot$ on the base AC is the maximum when the $\triangle$ is isosceles.

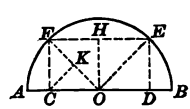
If now AB' and AC are unequal, considering $B'C$ the base of the $\triangle$, the $\triangle$ with the maximum perimeter is isosceles.

That is, the perimeter of the $\triangle ABC$ is the maximum when the $\triangle$ is equilateral. Q. E. D.

Ex. 449. *To inscribe in a semicircle the maximum rectangle.*

Let AB be the diameter and O the centre of the given semicircle.

To inscribe in the semicircle the maximum rectangle.



Analysis. Suppose the problem solved, and let $CDEF$ be the maximum rectangle required.

Draw the radii OE and OF.

Draw $OH \perp$ to EF, and $CK \perp$ to OF.

Now rt. $\triangle OCF$ = rt. $\triangle FHO$ = rt. $\triangle EHO$ = rt. $\triangle ODE$. § 151

For $OF = OF = OE = OE$, § 217

and $CF = HO = HO = DE$. §§ 104, 180

$\therefore \triangle OCF = \frac{1}{4}$ the rectangle $CDEF$.

That the rectangle $CDEF$ may be the maximum, the rt. $\triangle OCF$ must be the maximum.

Since the hypotenuse OF of the rt. $\triangle OCF$ is determined, that the rt. $\triangle OCF$ may be the maximum, CK must be the maximum.

But CK is the maximum when $KO = KF$; that is, when $\angle FOC = 45°$.

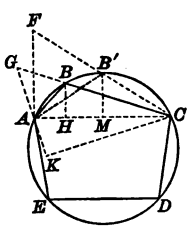
Construction. Draw the radii OE and OF, making angles of 45° with AB. Draw ED and $FC \perp$ to AB, and draw EF.

The rectangle $CDEF$ is the maximum rectangle required. Q. E. F.

Ex. 450. *Of all polygons of a given number of sides which may be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.*

Let $ABCDE$ be a polygon of n sides inscribed in the given $\odot AED$.

1. To prove that $ABCDE$ is the maximum when regular.



Proof. If $ABCDE$ is not a regular polygon, at least two adjacent sides are unequal.

Let BC be greater than AB.

Draw AC and draw $BH \perp$ to AC.

Erect MB', the $\perp$ bisector of AC, meeting the circumference at B', and draw AB' and CB'.

Since MB' passes through the centre of the $\odot$, § 248

the tangent through B' is $\parallel$ to AC. § 104

B' lies in this tangent and B lies between the tangent and AC. § 220

$\therefore B'M > BH.$

$\therefore \triangle AB'C > \triangle ABC.$ § 405

$\therefore$ polygon $AB'CDE >$ polygon $ABCDE.$

But $AB' = B'C.$ § 160

Therefore, each $\triangle$ constructed in a similar manner to $\triangle ABC$ is the maximum when isosceles.

That is, the polygon $ABCDE$ is the maximum when equilateral.

But an equilateral polygon inscribed in a $\odot$ is regular. § 430

Therefore, the polygon $ABCDE$ is the maximum when regular.

2. To prove that the perimeter of the polygon is the maximum when the polygon is regular.

Proof. Produce CB' to F, making $B'F$ equal to $B'A$, and produce CB to G, making BG equal to BA. Draw AF and AG.

Draw $CK \perp$ to GA produced.

Now FA is $\perp$ to AC. Ex. 43

$\angle F = \angle B'AF$, and $\angle G = \angle BAG$. § 145

$\therefore \angle AB'C = 2\angle F$, and $\angle ABC = 2\angle G$. § 137

Now $\angle AB'C = \angle ABC$. § 293

$\therefore \angle F = \angle G$. Ax. 7

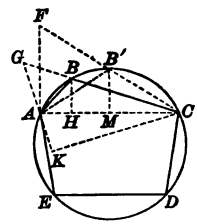
$\therefore$ the rt. $\triangle FAC$ and GKC are similar. § 356

$\therefore CF : CG = CA : CK$. § 351

That is, $CB' + B'A : CB + BA = CA : CK$.

But $CA > CK$. § 97

$\therefore CB' + B'A > CB + BA$.



Therefore, the perimeter of the polygon $AB'CDE$ is greater than the perimeter of the polygon $ABCDE$.

That is, the perimeter of the polygon is the maximum when the polygon is equilateral.

But an equilateral polygon inscribed in a $\odot$ is regular. § 430

Therefore, the perimeter of the polygon is the maximum when the polygon is regular. Q. E. D.

Ex. 451. *Of all polygons of a given number of sides which may be circumscribed about a given circle that which is regular has the minimum area and the minimum perimeter.*

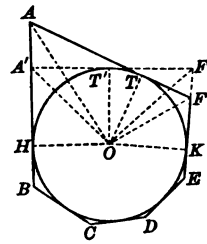
Let $ABCDEF$ be a polygon of n sides circumscribed about a given $\odot$ whose centre is O.

1. To prove that $ABCDEF$ is the minimum when regular.

Proof. If $ABCDEF$ is not a regular polygon, at least two adjacent arcs between the points of tangency are unequal.

Suppose the arc HT to be greater than the arc TK.

Through T', the middle point of the arc HK, draw the tangent $A'F'$ intersecting the side AB in A' and the side EF produced in F'; draw OH, OA', OA, OT', OT, OF', OF, OK.



The rt. $\triangle OHA'$ and $OT'A'$ are equal. § 144

For $OH = OT'$, § 217

and $OA' = OA'$. Iden.

$\therefore \angle A'OT' = \frac{1}{2}\angle HOT'$. § 128

Similarly, $\angle T'OF' = \frac{1}{2}\angle T'OK$.

But $\angle HOT' = \angle T'OK$. Const.

$\therefore \angle A'OT' = \angle T'OF'$.

$\therefore \angle A'OF' = \frac{1}{2}\angle HOK$.

$\therefore \angle T'A'O = \angle T'F'O.$ § 84

$\therefore OA' = OF'.$ § 147

$\therefore \triangle A'OF'$ is isosceles.

In a similar manner, $\triangle AOF$ is not isosceles, since $\angle HOT$ is greater than $\angle TOK$, but $\angle AOF = \frac{1}{2}\angle HOK = \angle A'OF'$.

$\therefore$ the $\measuredangle$ $A'OF'$ and AOF are $\measuredangle$ having equal altitudes OT' and OT and equal vertical $\measuredangle$ $A'OF'$ and AOF.

$\therefore \triangle AOF > \triangle A'OF'$, since $AF > A'F'$.

Now $A'HOKF' = 2\triangle A'OF'$, and $AHOKF = 2\triangle AOF$.

$\therefore A'HOKF' < AHOKF.$ Ax. 6

$\therefore A'BCDEF' < ABCDEF.$ Ax. 4

$\therefore$ the circumscribed polygon is not the minimum when any two adjacent arcs formed by the points of tangency are unequal.

But if all these arcs are equal, the polygon is regular. § 439

$\therefore$ the polygon is the minimum when regular.

2. To prove that $ABCDEF$ has the minimum perimeter when regular.

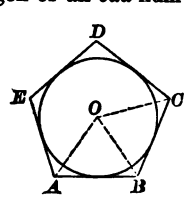
$A'F'$ has been proved to be less than AF.

That is, as long as the circumscribed polygon is not regular its perimeter is not the minimum.

$\therefore$ the perimeter of the circumscribed polygon is the minimum when the polygon is regular. Q. E. D.

Ex. 452. *Every equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.*

Let $ABCDE$ be an equilateral circumscribed polygon of an *odd* number of sides.



To prove that $ABCDE$ is a regular polygon.

Proof. Join the centre O with A, B, C.

Now $\triangle OBA = \triangle OBC.$ § 143

For $AB = BC,$ Hyp.

$\angle OBA = \angle OBC,$ § 261

and $OB = OB.$ Iden.

$\therefore \angle OAB = \angle OCB.$ § 128

$\therefore \angle BAE = \angle BCD.$ Ax. 6

In like manner, it may be proved that the first of every three consecutive $\measuredangle$ is equal to the third.

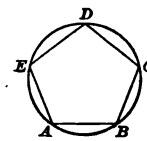
That is, $\angle A = \angle C = \angle E = \angle B = \angle D.$

$\therefore$ the polygon $ABCDE$ is a regular polygon, being equilateral and equiangular. § 429

The same method is manifestly applicable to any polygon of an odd number of sides. Q. E. D.

Ex. 453. *Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides.*

Let $ABCDE$ be an inscribed equiangular polygon having an *odd* number of sides.



To prove that $ABCDE$ is a regular polygon.

Proof. Since the equal inscribed $\measuredangle$ intercept equal arcs on the circumference, if each arc is subtracted from the circumference, the remainders are equal.

That is, arc ABC = arc BCD.

Take the arc BC from each of these equals.

Then arc AB = arc CD. Ax. 3

In like manner, arc CD = arc EA = arc BC = arc DE = arc AB.

$\therefore AB = BC = CD$, etc. § 241

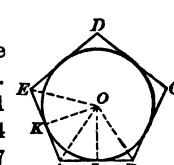
Hence, the polygon $ABCDE$ is a regular polygon, being equilateral and equiangular. § 429

The same method is manifestly applicable to any polygon of an odd number of sides. Q. E. D.

Ex. 454. *Every equiangular polygon circumscribed about a circle is regular.*

Let $ABCDE$ be an equiangular polygon circumscribed about the $\odot$ whose centre is O.

To prove that $ABCDE$ is a regular polygon.



Proof. Draw OA, OB, OE. Draw OK and OL to the points of contact of the sides AE and AB, respectively.

$\triangle OAK = \triangle OAL$. § 151

For $\angle OKA = \angle OLA = 90°$, § 254

$OA = OA$ (Iden.), and $OK = OL$. § 217

$\therefore AK = AL$. § 128

Now $\angle OAL = \angle OBL$ (being halves of the equal $\measuredangle$ A and B). Ax. 7

$\therefore \triangle OLA = \triangle OLB$. § 142

$\therefore LA = LB$. § 128

$\therefore AB = 2\,AL$.

In like manner, $AE = 2\,AK$.

But $AL = AK$.

$\therefore AB = AE$. Ax. 6

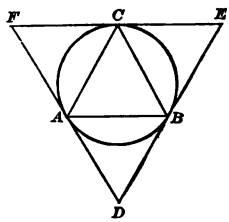
In like manner, $AB = BC = CD = DE = EA$.

Hence, the polygon $ABCDE$ is a regular polygon, being equilateral and equiangular. § 429

Q. E. D.

Ex. 455. *The side of a circumscribed equilateral triangle is equal to twice the side of the similar inscribed triangle.*

Let DEF be a circumscribed equilateral $\triangle$, and ABC an inscribed equilateral $\triangle$.



To prove that $EF = 2\,CB$.

Proof.

Now	$EB = EC$.	§ 261
	$\therefore \angle ECB = \angle EBC$.	§ 145
But	$\angle E = 60°$.	Hyp.
	$\therefore \angle EBC = \angle ECB = 60° = \angle E$.	
	$\therefore EC = CB$.	§ 147
In like manner,	$FC = AC = CB$.	
	$\therefore EF = 2\,CB$.	Q. E. D.

Ex. 456. *The apothem of an inscribed regular hexagon is equal to half the side of the inscribed equilateral triangle.*

Proof.

The apothem of the hexagon	$= \frac{1}{2} R\sqrt{3}$.	Ex. 442
The side of the $\triangle$	$= R\sqrt{3}$.	Ex. 440

Q. E. D.

Ex. 457. *The area of an inscribed regular hexagon is three fourths of the area of the circumscribed regular hexagon.*

Proof. The apothem of the circumscribed hexagon = radius R.

The apothem of the inscribed hexagon $= \frac{1}{2} R\sqrt{3}$. Ex. 442

$$\therefore \frac{\text{area inscribed hexagon}}{\text{area circumscribed hexagon}} = \frac{(\frac{1}{2} R\sqrt{3})^2}{R^2} = \frac{3\,R^2}{4\,R^2} = \frac{3}{4}. \quad \text{§ 448}$$

Q. E. D.

Ex. 458. *The area of an inscribed regular hexagon is the mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.*

Proof. The area of the inscribed equilateral $\triangle$ whose side is a

$$= \frac{a}{2} \times \tfrac{3}{2} R = \frac{R\sqrt{3}}{2} \times \frac{3\,R}{2} = \tfrac{3}{4} R^2\sqrt{3}. \quad \text{Ex. 440}$$

The area of the circumscribed equilateral $\triangle$

$$= 4 \times \tfrac{3}{4} R^2\sqrt{3} = 3\,R^2\sqrt{3}. \quad \text{Ex. 455}$$

The mean proportional between these areas

$$= \sqrt{\tfrac{3}{4} R^2\sqrt{3} \times 3\,R^2\sqrt{3}} = \tfrac{3}{2} R^2\sqrt{3}.$$

The area of a regular inscribed hexagon

$$= \frac{6\,R}{2} \times \tfrac{1}{2} R\sqrt{3} = \tfrac{3}{2} R^2\sqrt{3}. \quad \text{Ex. 442}$$

Q. E. D.

Ex. 459. *The square of the side of an inscribed equilateral triangle is equal to three times the square of a side of the inscribed regular hexagon.*

Proof. The side of an inscribed equilateral $\triangle = R\sqrt{3}$, Ex. 440
and the square of the side $= 3R^2$.
The side of an inscribed regular hexagon $= R$, Ex. 442
and the square of the side $= R^2$. Q. E. D.

Ex. 460. *The area of an inscribed equilateral triangle is equal to half the area of the inscribed regular hexagon.*

Area of an inscribed equilateral triangle $= \frac{3}{4}R^2\sqrt{3}$. Ex. 458
Area of an inscribed regular hexagon $= \frac{3}{2}R^2\sqrt{3}$. Ex. 458
Now $\frac{3}{4}R^2\sqrt{3}$ = half of $\frac{3}{2}R^2\sqrt{3}$. Q. E. D.

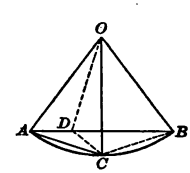
Ex. 461. *The square of the side of an inscribed equilateral triangle is equal to the sum of the squares of the sides of the inscribed square and of the inscribed regular hexagon.*

The side of an inscribed equilateral triangle $= R\sqrt{3}$. Ex. 440
The side of an inscribed square $= R\sqrt{2}$. Ex. 441
The side of an inscribed regular hexagon $= R$. Ex. 442
Now $(R\sqrt{3})^2 = 3R^2$,
and $(R\sqrt{2})^2 + R^2 = 2R^2 + R^2 = 3R^2$. Q. E. D.

Ex. 462. *The square of the side of an inscribed regular pentagon is equal to the sum of the squares of the radius of the circle and the side of the inscribed regular decagon.*

Let AB be the side of a regular inscribed pentagon, and AC the side of a regular inscribed decagon.

To prove that $\overline{AB}^2 = \overline{OB}^2 + \overline{AC}^2$.



Proof. Draw OD bisecting the $\angle AOC$ and meeting AB in D. Draw DC and BC.

Now $\triangle OAD = \triangle OCD$. § 143
For $OA = OC$, § 217
$\angle AOD = \angle COD$, Const.
and $OD = OD$. Iden.
$\therefore AD = DC$. § 128
$\therefore$ △ADC and BCA are similar. Ex. 264
$\therefore AC : AB = AD : AC$. § 351
$\therefore \overline{AC}^2 = AB \times AD$. § 327
Again, $\angle AOB = \frac{1}{5}$ of $360° = 72°$, § 436
and $\angle AOC = \frac{1}{2}$ of $72° = 36°$.

$\therefore \angle COD = \frac{1}{2}$ of $36° = 18°$.

$\therefore \angle DOB = 36° + 18° = 54°$.

But $\angle OBA = \frac{1}{2}(180° - AOB)$

$= \frac{1}{2}(180° - 72°) = 54°$.

That is, the $\triangle DOB$ is isosceles and similar to the $\triangle OAB$, having the $\angle OBD$ in common.

$\therefore AB : OB = OB : BD.$ § 351

$\therefore \overline{OB}^2 = AB \times BD.$ § 327

$\therefore \overline{OB}^2 + \overline{AC}^2 = AB\,(BD + AD) = AB \times AB = \overline{AB}^2.$ Ax. 2

Q. E. D.

Ex. 463. *If R denotes the radius of a circle, and a one side of a regular inscribed pentagon, show that*

$$a = \tfrac{1}{2}R\sqrt{10 - 2\sqrt{5}}.$$

Proof. The side of an inscribed decagon

$$= \frac{R(\sqrt{5} - 1)}{2}.$$ Ex. 443

$$\therefore a^2 = R^2 + \left\{\frac{R(\sqrt{5} - 1)}{2}\right\}^2$$ Ex. 462

$$= R^2 + \frac{R^2(6 - 2\sqrt{5})}{4}$$

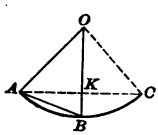
$$= \frac{4R^2 + R^2(6 - 2\sqrt{5})}{4}$$

$$= \frac{R^2(10 - 2\sqrt{5})}{4}.$$

$$\therefore a = \tfrac{1}{2}R\sqrt{10 - 2\sqrt{5}}.$$ Q. E. D.

Ex. 464. *If R denotes the radius of a circle, and a one side of a regular inscribed octagon, show that*

$$a = R\sqrt{2 - \sqrt{2}}.$$



Let AB be the side of a regular inscribed octagon, and R the radius.

To prove that $a = R\sqrt{2 - \sqrt{2}}$.

Proof. Draw AC, the side of an inscribed square.

$$OK = \tfrac{1}{2}R\sqrt{2}.$$ Ex. 441

$$BK = OB - OK = R - \tfrac{1}{2}R\sqrt{2} = \tfrac{1}{2}R(2 - \sqrt{2}).$$

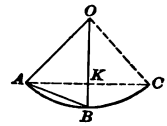
Now $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2.$ § 371

$\therefore a^2 = (\frac{1}{2}R\sqrt{2})^2 + \{\frac{1}{2}R(2-\sqrt{2}\}^2$ Ex. 441

$= \frac{1}{2}R^2 + \frac{1}{2}R^2(6-4\sqrt{2})$

$= R^2(2-\sqrt{2})$.

$\therefore a = R\sqrt{2-\sqrt{2}}$. Q. E. D.

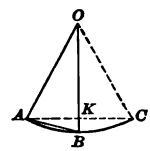


Ex. 465. *If R denotes the radius of a circle, a the side of a regular inscribed dodecagon, show that*

$$a = R\sqrt{2-\sqrt{3}}.$$

Let AB be the side of a regular inscribed dodecagon, and R the radius.

To prove that $a = R\sqrt{2-\sqrt{3}}$.



Proof. Draw AC, the side of a regular inscribed hexagon.

$OK = \frac{1}{2}R\sqrt{3}$. Ex. 442

$BK = OB - OK = R - \frac{1}{2}R\sqrt{3} = \frac{1}{2}R(2-\sqrt{3})$.

Now $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2$. § 371

$\therefore a^2 = (\frac{1}{2}R)^2 + \{\frac{1}{2}R(2-\sqrt{3})\}^2$ Ex. 442

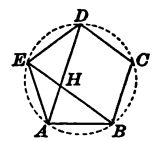
$= \frac{1}{4}R^2 + \frac{1}{4}R^2(7-4\sqrt{3})$

$= R^2(2-\sqrt{3})$.

$\therefore a = R\sqrt{2-\sqrt{3}}$. Q. E. D.

Ex. 466. *If two diagonals of a regular pentagon intersect, the longer segment of each is equal to a side of the pentagon.*

Let $ABCDE$ be a regular pentagon, and let AD and BE be two diagonals intersecting at H.



To prove that $BH = HD = AB$.

Proof. Circumscribe a ⊙ about the pentagon. § 431

Now arc AB = arc BC = arc CD = arc DE

= arc EA = 72°. § 243

$\angle AHB = 72°$, being measured by $\frac{1}{2}$ (arc AB + arc DE). § 294

$\angle DAB = 72°$, being measured by $\frac{1}{2}$ (arc BC + arc CD). § 289

$\therefore \angle AHB = \angle DAB$. Ax. 1

$\therefore AB = BH$. § 147

In like manner, $ED = HD$.

But $AB = ED$. Hyp.

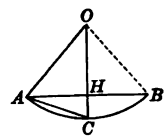
$\therefore BH = HD = AB$. Ax. 1

Q. E. D.

Ex. 467. *The apothem of an inscribed regular pentagon is equal to half the sum of the radius of the circle and the side of the inscribed regular decagon.*

Let AB be the side of an inscribed regular pentagon, and AC the side of an inscribed regular decagon.

To prove that $OH = \frac{1}{2}(OA + AC)$.



Proof.

$AB = \frac{1}{2}R\sqrt{10 - 2\sqrt{5}}$. Ex. 463

$\therefore AH = \frac{1}{2}R\sqrt{10 - 2\sqrt{5}}$.

$AC = \frac{1}{2}R(\sqrt{5} - 1)$. Ex. 443

$OA = R$.

Now $\overline{OH}^2 = \overline{OA}^2 - \overline{AH}^2$. § 372

$\therefore \overline{OH}^2 = R^2 - (\frac{1}{2}R\sqrt{10 - 2\sqrt{5}})^2$

$= R^2 - \frac{1}{16}R^2(10 - 2\sqrt{5})$

$= \frac{1}{16}R^2(6 + 2\sqrt{5})$.

$\therefore OH = \frac{1}{4}R\sqrt{6 + 2\sqrt{5}}$

$= \frac{1}{4}R\sqrt{1 + 2\sqrt{5} + 5}$

$= \frac{1}{4}R(1 + \sqrt{5})$.

$\frac{1}{2}(OA + AC) = \frac{1}{2}\{R + \frac{1}{4}R(\sqrt{5} - 1)\}$

$= \frac{1}{2}(\frac{1}{4}R + \frac{1}{4}R\sqrt{5})$

$= \frac{1}{4}R(1 + \sqrt{5})$.

$\therefore OH = \frac{1}{2}(OA + AC)$. Ax. 1

Q. E. D.

Ex. 468. *The side of an inscribed regular pentagon is equal to the hypotenuse of the right triangle which has for legs the radius of the circle and the side of the inscribed regular decagon.*

Let AB be the side of an inscribed regular pentagon and AC the side of an inscribed regular decagon.

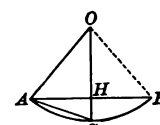
To prove that $\overline{AB}^2 = \overline{OA}^2 + \overline{AC}^2$.

Proof.

$AB = \frac{1}{2}R\sqrt{10 - 2\sqrt{5}}$. Ex. 463

$OA = R$.

$AC = \frac{1}{4}R(\sqrt{5} - 1)$. Ex. 443



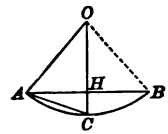
$\therefore \overline{OA}^2 + \overline{AC}^2 = R^2 + \{\frac{1}{4}R(\sqrt{5} - 1)\}^2$

$= R^2 + \frac{1}{8}R^2(6 - 2\sqrt{5})$

$= R^2 + \frac{1}{4}R^2(3 - \sqrt{5})$

$= \frac{1}{4}R^2(5 - \sqrt{5})$.

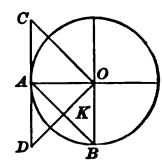
$$\overline{AB}^2 = (\tfrac{1}{2}R\sqrt{10-2\sqrt{5}})^2$$
$$= \tfrac{1}{4}R^2(10-2\sqrt{5})$$
$$= \tfrac{1}{4}R^2(5-\sqrt{5}).$$
$$\therefore \overline{AB}^2 = \overline{OA}^2 + \overline{AC}^2. \qquad \text{Ax. 1}$$



Q. E. D.

Ex. 469. *The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of the similar circumscribed regular polygon.*

Let OA be the radius and OK the apothem of the inscribed regular polygon, of which AB is a side. Let OD be the radius of the similar circumscribed regular polygon of which CD is a side.



To prove that $OD : OA = OA : OK$.

Proof. Since CD is tangent to the $\odot$ at A, the $\angle OAD$ is a rt. $\angle$. § 254

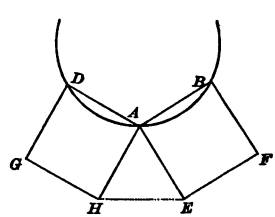
But AK is $\perp$ to OD. Hyp.

$\therefore OD : OA = OA : OK$. § 367

Q. E. D.

Ex. 470. *If squares are constructed outwardly upon the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.*

Let DA and AB be adjacent sides of a regular inscribed hexagon, and $ADGH$ and $ABFE$ be squares constructed outwardly on AD and AB.



To prove that G, H, E, F, etc., are the vertices of a regular dodecagon.

Proof. Draw HE.

$\angle DAB = 120°$. Ex. 442

$\angle HAE = 360° - (DAB + DAH + BAE) = 360° - (120° + 90° + 90°) = 60°$.

Also, $AH = AE$, since they are sides of equal squares.

$\therefore \angle AHE = \angle AEH$. § 145

But $\angle HAE = 60°$.

$\therefore \angle AHE = \angle AEH = 60°$.

$\therefore AE = EH$. § 147

$\therefore GH = HE = EF$, etc.

That is, the dodecagon is equilateral.

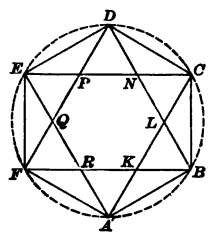
Now, $\angle GHE = \angle HEF$, etc., since each is the sum of a right angle and an angle of 60°. That is, the decagon is equiangular.

Hence, G, H, E, F, etc., are the vertices of a regular dodecagon. § 429

Q. E. D.

Ex. 471. *If the alternate vertices of a regular hexagon are joined by straight lines, show that another regular hexagon is thereby formed. Find the ratio of the areas of these two hexagons.*

Let $ABCDEF$ be a regular hexagon. Let BD, CE, etc., join the alternate vertices and intersect at N, P, Q, R, K, and L.



To prove that $NPQRKL$ is a regular hexagon.

Proof. Circumscribe a ⊙ about the hexagon $ABCDEF$. § 431

arc AB = arc BC = arc CD = arc DE = arc EF = arc FA = 60°. § 243

$\angle AEC = 60°$, being measured by $\frac{1}{2}$(arc AB + arc BC). § 289

$\angle EPQ = 60°$, being measured by $\frac{1}{2}$(arc EF + arc DC). § 294

$\angle EQP = 60°$, being measured by $\frac{1}{2}$ (arc ED + arc FA). § 294

Hence, $\triangle EPQ$ is equilateral. § 148

In like manner, $\triangle FQR$, ARK, etc., are equilateral.

$\angle QEF = \angle QFE$, being measured by half the equal arcs FA and DE. § 289

$\therefore QE = QF$. § 147

But $QE = QP$, and $QF = QR$.

$\therefore QP = QR$. Ax. 1

In like manner, $QR = RK = KL$, etc.

That is, $NPQRKL$ is equilateral.

Now $\angle NPQ = \angle PQR = \angle QRK$, etc. § 85

That is, $NPQRKL$ is equiangular.

Hence, $NPQRKL$ is regular. § 429

Now it has been shown that $DP = PQ = QF$.

$\therefore PQ = \frac{1}{3}DF = \frac{1}{3}R\sqrt{3}$. Ex. 440

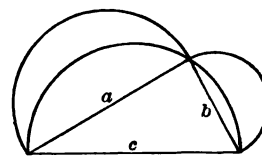
But $DE = R$. § 469

$$\therefore \frac{ABCDEF}{NPQRKL} = \frac{R^2}{(\frac{1}{3}R\sqrt{3})^2} = 3. \quad \text{§ 446}$$

That is, the area of the outer hexagon is 3 times the area of the inner hexagon.

Q. E. D.

Ex. 472. *If on the legs of a right triangle as diameters semicircles are described external to the triangle, and from the whole figure a semicircle on the hypotenuse is subtracted, the remaining figure is equivalent to the given right triangle.*



Let a and b be the legs, and c the hypotenuse of the rt. $\triangle$.

To prove that the semicircles on a and b as diameters, increased by the area of the $\triangle$ and diminished by the semicircle on c as a diameter, are equivalent to the area of the $\triangle$.

Proof. The sum of the areas of the semicircles on a and b as diameters

$= \frac{1}{8}\pi a^2 + \frac{1}{8}\pi b^2$ § 463

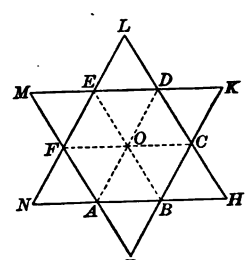
$= \frac{1}{8}\pi (a^2 + b^2)$

$= \frac{1}{8}\pi c^2$. § 371

But the area of the semicircle on c as diameter $= \frac{1}{8}\pi c^2$. § 463

Now, if the area of the $\triangle$ is increased by $\frac{1}{8}\pi c^2$ and then diminished by $\frac{1}{8}\pi c^2$, the area of the triangle will remain. Q. E. D.

Ex. 473. *The star-shaped polygon, formed by producing the sides of a regular hexagon, is equivalent to twice the given hexagon.*



Let $ABCDEF$ be a regular hexagon. Produce the sides of the hexagon both ways until they meet at the points H, K, L, M, N, P.

To prove that the star-shaped polygon $HKLMNP$ = twice the hexagon $ABCDEF$.

Proof. From the centre O of the hexagon draw the radii OA, OB, OC, OD, OE, OF.

$\angle AOB = \frac{1}{6}$ of $360° = 60°$. § 436

$\angle FAB = \angle ABC = 120°$. § 438

$\therefore \angle OAB = \angle OBA = 60°$. § 437

$\therefore \triangle AOB$ is equilateral. § 148

Now $\angle BAP = \angle ABP = 60°$. § 85

$\therefore \angle APB = 60°$. § 131

$\therefore \triangle PAB$ is equilateral. § 148

$\therefore \triangle PAB = \triangle AOB$. § 139

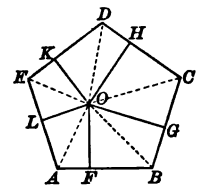
In like manner, $\triangle HBC = \triangle BOC$, etc.

But the sum of the $\triangle$s PAB, AOB, HBC, BOC, etc., is the star-shaped polygon $HKLMNP$, and the sum of the $\triangle$s AOB, BOC, etc., is the regular hexagon $ABCDEF$.

$\therefore$ the star-shaped polygon = twice the regular hexagon. Q. E. D.

Ex. 474. *The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the apothem multiplied by the number of sides.*

Let $ABCDE$ be a regular polygon of n sides. Let OF, OG, OH, OK, OL be $\perp$s drawn from any point O within the polygon to the sides. Let r denote the apothem, and P the perimeter of the polygon.



To prove that $OF + OG + OH + OK + OL = nr$.

Proof. Join O to each of the vertices of the polygon.

The area of the $\triangle AOB = \frac{1}{2} AB \times OF$, § 403

the area of the $\triangle BOC = \frac{1}{2} BC \times OG$,

the area of the $\triangle COD = \frac{1}{2} CD \times OH$, etc.

But $AB = BC = CD$, etc. Hyp.

Now the area of the polygon is the sum of the areas of the $\triangle AOB$, BOC, COD, etc.

$\therefore$ the area of the polygon $= \frac{1}{2} AB \times (OF + OG + OH + OK + OL)$.

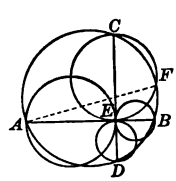
But the area of the polygon $= \frac{1}{2} P \times r = n \times \frac{1}{2} AB \times r$. § 459

$\therefore \frac{1}{2} AB \times (OF + OG + OH + OK + OL) = n \times \frac{1}{2} AB \times r$. Ax. 1

$\therefore OF + OG + OH + OK + OL = nr$. Q. E. D.

Ex. 475. *If two chords of a circle are perpendicular to each other, the sum of the four circles described on the four segments as diameters is equivalent to the given circle.*

Let the chords AB and CD be $\perp$ to each other at E. Let Ⓢ be described on AE, EB, CE, ED as diameters.



To prove that

$\odot AE + \odot EB + \odot CE + \odot ED \sim$ the given $\odot$.

Proof. Draw the diameter AF of the given $\odot$.

The area of any $\odot = \pi R^2 = \frac{1}{4} \pi D^2$. § 463

Therefore, the sum of the areas of the Ⓢ described on the segments $= \frac{1}{4} \pi \overline{AE}^2 + \frac{1}{4} \pi \overline{EB}^2 + \frac{1}{4} \pi \overline{CE}^2 + \frac{1}{4} \pi \overline{ED}^2$

$= \frac{1}{4} \pi (\overline{AE}^2 + \overline{EB}^2 + \overline{CE}^2 + \overline{ED}^2)$.

But $\overline{AE}^2 + \overline{EB}^2 + \overline{CE}^2 + \overline{ED}^2 = \overline{AF}^2$. Ex. 267

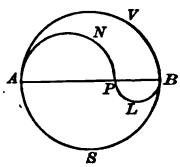
$\therefore \frac{1}{4} \pi (\overline{AE}^2 + \overline{EB}^2 + \overline{CE}^2 + \overline{ED}^2) = \frac{1}{4} \pi \overline{AF}^2$.

But the area of the $\odot AF = \frac{1}{4} \pi \overline{AF}^2$. § 463

$\therefore \odot AE + \odot EB + \odot CE + \odot ED \sim \odot AF$. Ax. 1

Q. E. D.

Ex. 476. *If the diameter of a circle is divided into any two segments, and upon these segments as diameters semicircumferences are described upon opposite sides of the diameter, these semicircumferences divide the circle into two parts which have the same ratio as the two segments of the diameter.*



Let AB be the diameter of the ⊙. Let P be any point on AB, at distance x from the centre, so that $AP = R + x$ and $PB = R - x$; upon AP and PB as diameters let semicircles be described on opposite sides of AB.

To prove that $\dfrac{\text{area } ANPLBS}{\text{area } AVBLPN} = \dfrac{R+x}{R-x}$.

Proof. The area of any circle $= \pi R^2 = \frac{1}{4}\pi D^2$. § 463

The area of a semicircle $= \frac{1}{8}\pi D^2$.

$$\frac{\text{area } ANPLBS}{\text{area } AVBLPN} = \frac{\text{semicircle } ANP + \text{semicircle } ASB - \text{semicircle } PLB}{\text{semicircle } AVB + \text{semicircle } PLB - \text{semicircle } ANP}$$

$$= \frac{\frac{1}{8}\pi(R+x)^2 + \frac{1}{2}\pi R^2 - \frac{1}{8}\pi(R-x)^2}{\frac{1}{2}\pi R^2 + \frac{1}{8}\pi(R-x)^2 - \frac{1}{8}\pi(R+x)^2}$$

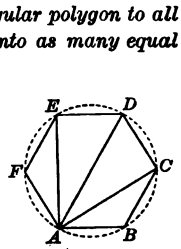
$$= \frac{(R+x)^2 + 4R^2 - (R-x)^2}{4R^2 + (R-x)^2 - (R+x)^2}$$

$$= \frac{4R^2 + 4Rx}{4R^2 - 4Rx}$$

$$= \frac{R+x}{R-x}.$$

Q. E. D.

Ex. 477. *The diagonals that join any vertex of a regular polygon to all the vertices not adjacent divide the angle at that vertex into as many equal parts less two as the polygon has sides.*



Let $ABCDEF$ be a regular polygon of n sides. Let all the possible diagonals from the vertex A be drawn.

To prove that $\angle FAB$ is divided into $n - 2$ equal parts.

Proof. From any vertex of a polygon of n sides there can evidently be drawn $n - 3$ diagonals which will divide the angle at that vertex into $n - 2$ parts.

Circumscribe a ⊙ about the polygon.

The arcs AB, BC, CD, DE, EF, FA are equal. § 243

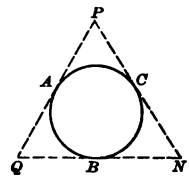
Each of the $n - 2$ angles at A is measured by half of one of these equal arcs. § 289

Therefore, the $n - 2$ angles are equal. Q. E. D.

Ex. 478. *To circumscribe an equilateral triangle about a given circle.*

Let ABC be a ⊙.

To circumscribe an equilateral △ about this ⊙.



Construction. Find the points A, B, C, the vertices of an inscribed equilateral △. § 470

Through the points A, B, C draw tangents (§ 317) intersecting at Q, N, P.

Then QNP is the △ required.

Proof. arc AB = arc BC = arc AC = 120°.

$\angle P$ is measured by ½ (arc AB + arc BC − arc AC) = ½ arc AC. § 296

$\therefore \angle P = 60°$.

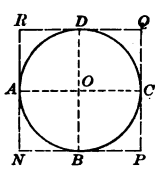
In like manner, $\angle N = 60° = \angle Q$.

$\therefore$ △ QNP is equilateral. § 148

Q. E. F.

Ex. 479. *To circumscribe a square about a given circle.*

Let ABC be a ⊙.



To circumscribe a square about this ⊙.

Construction. Divide the circumference into four equal arcs at the points A, B, C, D. § 467

Through the points of division draw tangents intersecting at N, P, Q, R.

Then $NPQR$ is the square required.

Proof. The $\angle P$ is measured by ½ (arc CDB − arc BC); § 296

that is, by ½ of ½ the circumference, or ¼ of the circumference.

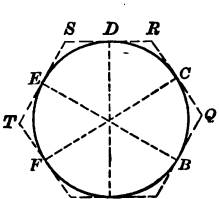
In like manner, the $\measuredangle Q$, R, N are each measured by ¼ of the circumference.

Hence, the quadrilateral is equiangular, and therefore regular. Ex. 454

That is, the regular quadrilateral $NPQR$ is a square. Q. E. F.

Ex. 480. *To circumscribe a regular hexagon about a given circle.*

Let ABC be a ⊙.

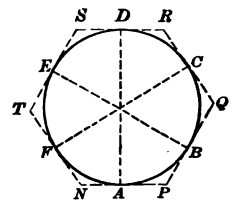


To circumscribe a regular hexagon about this ⊙.

Construction. Divide the circumference into six equal arcs at the points A, B, C, D, E, F. § 469

Through the points of division draw tangents intersecting at the points N, P, Q, R, S, T.

Then $NPQRST$ is the regular hexagon required.



Proof. The $\angle P$ is measured by

$\frac{1}{2}$(arc AEB − arc AB); § 296

that is, by $\frac{1}{2}$ of $\frac{6}{8}$ of the circumference, or $\frac{3}{8}$ of the circumference.

In like manner, the $\measuredangle Q$, R, S, T, N are each measured by $\frac{3}{8}$ of the circumference.

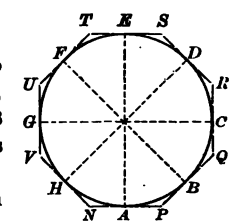
Hence, the hexagon is equiangular, and therefore regular. Ex. 454

Q. E. F.

Ex. 481. *To circumscribe a regular octagon about a given circle.*

Let ABC be a circle.

To circumscribe a regular octagon about this ⊙.



Construction. Divide the circumference into eight equal arcs at the points A, B, C, D, E, F, G, H. § 468

Through the points of division draw tangents intersecting at N, P, Q, R, S, T, U, V.

Then $NPQRSTUV$ is the regular octagon required.

Proof. The $\angle P$ is measured by $\frac{1}{2}$ (arc AEB − arc AB); § 296

that is, by $\frac{1}{2}$ of $\frac{6}{8}$ of the circumference, or $\frac{3}{8}$ of the circumference.

In like manner, the $\measuredangle Q$, R, S, T, U, V, N are each measured by $\frac{3}{8}$ of the circumference.

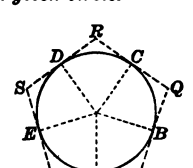
Hence, the octagon is equiangular, and therefore regular. Ex. 454

Q. E. F.

Ex. 482. *To circumscribe a regular pentagon about a given circle.*

Let ABC be a ⊙.

To circumscribe a regular pentagon about this ⊙.



Construction. Divide the circumference into five equal arcs at the points A, B, C, D, E. § 473

Through the points of division draw tangents intersecting at N, P, Q, R, S.

Then $NPQRS$ is the regular pentagon required.

Proof. The $\angle P$ is measured by $\frac{1}{2}$ (arc ADB − arc AB); § 296

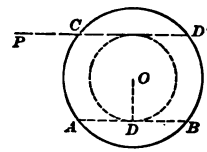
that is, by $\frac{1}{2}$ of $\frac{3}{5}$ of the circumference, or $\frac{3}{10}$ of the circumference.

In like manner, the $\measuredangle Q$, R, S, N are each measured by $\frac{3}{10}$ of the circumference.

Hence, the pentagon is equiangular, and therefore regular. Ex. 454

Q. E. F.

Ex. 483. *To draw through a given point a line so as to divide a given circumference into two parts having the ratio* 3 : 7.



Let ABD' be the given ⊙ and P the given point.

To draw a line through P which shall divide the circumference into two arcs having the ratio 3 : 7.

Construction. Divide the circumference into 10 equal parts. § 472

Let the arc AB include three of these parts.

Draw AB and from the centre O let fall $OD \perp$ to AB.

With O as centre and OD as radius describe a ⊙.

From the point P draw PCD' tangent to this ⊙. § 317

Then PD' divides the circumference as required.

Proof. chord CD' = chord AB. § 249

$\therefore$ arc CD' = arc AB. § 240

But arc $AB = \frac{3}{10}$ of the circumference. Const.

$\therefore$ arc $CD' = \frac{3}{10}$ of the circumference.

$\therefore$ arc CD' : arc $CAD' = 3 : 7$. Q. E. F.

Ex. 484. *To construct a circumference equal to the sum of two given circumferences.*

Let C and C' be two circumferences with radii R and R', respectively.

To construct a circumference equal to $C + C'$.

Analysis. $C = 2\pi R$, and $C' = 2\pi R'$. § 458

$\therefore C + C' = 2\pi(R + R')$.

Construction. With $R + R'$ as a radius describe a circumference.

This circumference is the circumference required.

Proof. The circumference whose radius is $R + R'$ is $2\pi(R + R')$. § 458

That is, the circumference $= 2\pi R + 2\pi R' = C + C'$. Q. E. F.

Ex. 485. *To construct a circumference equal to the difference of two given circumferences.*

Let C and C' be two given circumferences whose radii are respectively R and R'. Let C be greater than C'.

To construct a circumference equal to $C - C'$.

Analysis. $C = 2\pi R$, and $C' = 2\pi R'$. § 458

$\therefore C - C' = 2\pi(R - R')$.

Construction. With $R - R'$ as a radius describe a circumference.

This circumference is the circumference required.

Proof. The circumference whose radius is $R - R'$ is $2\pi(R - R')$. § 458

That is, the circumference $= 2\pi R - 2\pi R' = C - C'$. Q. E. F.

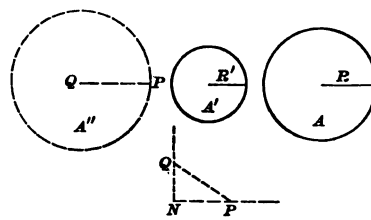
Ex. 486. *To construct a circle equivalent to the sum of two given circles.*

Let A and A' be two Ⓢ whose radii are R and R', respectively.

To construct a ⊙ equivalent to $A + A'$.

Construction. Construct the rt. $\angle QNP$, and take NP equal to R and NQ equal to R'.

Draw QP, and with QP as a radius construct a ⊙ A''.



Then A'' is the ⊙ required.

Proof. $A = \pi R^2$, $A' = \pi R'^2$, and $A'' = \pi \overline{QP}^2$. § 463

Now $\overline{QP}^2 = \overline{NP}^2 + \overline{NQ}^2$. § 371

$\therefore A'' = \pi \overline{QP}^2 = \pi (\overline{NP}^2 + \overline{NQ}^2) = \pi (R^2 + R'^2)$.

$\therefore A'' \doteq A + A'$. Q. E. F.

Ex. 487. *To construct a circle equivalent to the difference of two given circles.*

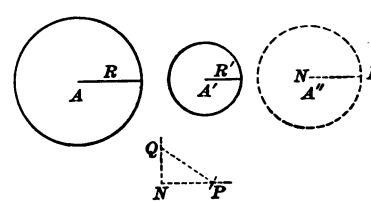
Let A and A' be two Ⓢ whose radii are R and R', respectively. Let A be greater than A'.

To construct a circle equivalent to $A - A'$.

Construction. Construct the rt. $\angle QNP$, and take NQ equal to R'.

With Q as a centre with a radius equal to R describe an arc intersecting NP at P.

With NP as a radius construct a ⊙ A''.



Then A'' is the ⊙ required.

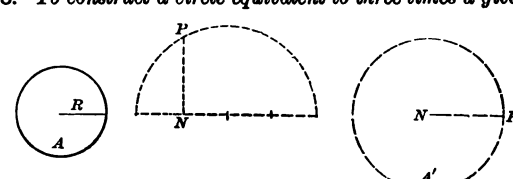
Proof. $A = \pi R^2$, $A' = \pi R'^2$, and $A'' = \pi \overline{NP}^2$. § 463

Now $\overline{NP}^2 = \overline{QP}^2 - \overline{NQ}^2$. § 372

$\therefore A'' = \pi \overline{NP}^2 = \pi (\overline{QP}^2 - \overline{NQ}^2) = \pi (R^2 - R'^2) = \pi R^2 - \pi R'^2$.

$\therefore A'' \doteq A - A'$. Q. E. F.

Ex. 488. *To construct a circle equivalent to three times a given circle.*



Let A be the given $\odot$, whose radius is R.

To construct a $\odot A'$ equivalent to $3A$.

Construction. With R as a unit construct NP equal to $\sqrt{3}$. Ex. 289

With NP as radius describe a $\odot A'$.

Then A' is the $\odot$ required.

Proof. $A = \pi R^2$. § 463

$\therefore A' = \pi \overline{NP}^2 = \pi (R\sqrt{3})^2 = 3\pi R^2$.

$\therefore A' = 3A$. Q. E. F.

Ex. 489. *To construct a circle equivalent to three fourths of a given circle.*

Let A be the given $\odot$, whose radius is R.

To construct a $\odot A'$ equivalent to $\frac{3}{4}A$.

Construction. With R as a unit construct NP equal to $\sqrt{3}$. Ex. 289

With $\frac{1}{2}NP$ as radius describe a $\odot A'$.

Then A' is the $\odot$ required.

Proof. $A = \pi R^2$. § 463

$A' = \pi (\frac{1}{2}NP)^2 = \pi (\frac{1}{2}R\sqrt{3})^2 = \frac{3}{4}\pi R^2$.

$\therefore A' = \frac{3}{4}A$. Q. E. F.

Ex. 490. *To construct a circle whose ratio to a given circle shall be equal to the given ratio $m : n$.*

Let A be the given $\odot$ whose radius is R.

To construct a $\odot A'$ which shall be to the $\odot A$ as $m : n$.

Construction. With R as a unit construct NP equal to $R\sqrt{\frac{m}{n}}$. Ex. 289

With NP as radius describe a $\odot A'$.

Then A' is the $\odot$ required.

Proof. $A = \pi R^2$. § 463

$$A' = \pi \overline{NP}^2 = \pi \left(R\sqrt{\frac{m}{n}}\right)^2 = \pi R^2 \times \frac{m}{n}.$$

$$\therefore A' : A = \pi R^2 \times \frac{m}{n} : \pi R^2$$

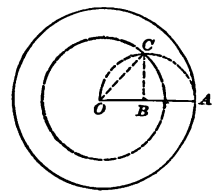
$$= \frac{m}{n} : 1$$

$$= m : n.$$ Q. E. F.

Ex. 491. *To divide a given circle by a concentric circumference into two equivalent parts.*

Let O be the centre and OA a radius of the given $\odot$.

To divide the ⊙ into two equivalent parts by a concentric circumference.



Construction. Bisect OA at B.

Erect the ⊥ BC, meeting at C the semicircumference OCA described on OA as a diameter, and draw OC.

With O as centre and OC as radius describe a circumference.

This circumference is the required circumference.

Proof.
$$\overline{OC}^2 = \overline{OB}^2 + \overline{BC}^2 \qquad \text{§ 371}$$
$$= 2\,\overline{OB}^2 = \frac{4\,\overline{OB}^2}{2} = \frac{\overline{OA}^2}{2}.$$
$$\therefore \pi\overline{OC}^2 = \frac{\pi\overline{OA}^2}{2}.$$

But $\pi\overline{OC}^2$ is the area of the inner ⊙, § 463

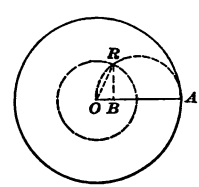
and $\dfrac{\pi\overline{OA}^2}{2}$ is ½ of the area of the given circle. Q. E. F.

Ex. 492. *To divide a circle by concentric circumferences into five equivalent parts.*

Let O be the centre and OA a radius of the given ⊙.

To divide the ⊙ into five equivalent parts by concentric circumferences.

Construction. Divide OA into five equal parts. § 307



At B, the first point of division from O, erect a ⊥, and let it meet the semicircumference on OA as a diameter at R.

With O as centre and OR as radius, describe a circumference.

This circumference includes ⅕ of the ⊙.

At K, the second point of division from O, erect a second ⊥.

With O as a centre, and a radius equal to the distance from O to the intersection of the ⊥ with the circumference, on OA as diameter describe a circumference.

This circumference includes ⅖ of the ⊙.

At the third and fourth points of division describe similar circumferences.

Proof.
$$\overline{OR}^2 = \overline{OB}^2 + \overline{BR}^2 \qquad \text{§ 371}$$
$$= \overline{OB}^2 + OB \times BA \qquad \text{§ 370}$$

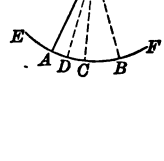
$$= OB(OB + BA)$$
$$= \tfrac{1}{5}\, OA \times OA = \tfrac{1}{5}\, \overline{OA}^2.$$
$$\therefore \pi\overline{OR}^2 = \tfrac{1}{5}\,\pi\overline{OA}^2.$$

But $\pi\overline{OR}^2$ is the area of the ⊙ of which OR is the radius, and $\pi\overline{OA}^2$ is the area of the given ⊙. § 463

In like manner, it may be shown that $\pi\overline{OK}^2 = \tfrac{2}{5}\pi\overline{OA}^2$, etc. Q. E. F.

Ex. 493. *To construct an angle of* 18°; *of* 36°; *of* 9°.

Let EF be an arc of a circle whose centre is O, and whose radius is OA.



1. To construct an angle of 36°.

Take arc AB equal to $\frac{1}{10}$ of the circumference. § 472

Draw OB.

$\angle AOB = \frac{1}{10}$ of 360° = 36°. § 288

2. To construct an angle of 18°.

Draw OC, bisecting the $\angle AOB$.

$\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2}$ of 36° = 18°.

3. To construct an angle of 9°.

Draw OD bisecting the $\angle AOC$.

$\angle AOD = \frac{1}{2}\angle AOC = \frac{1}{2}$ of 18° = 9°. Q. E. F.

Ex. 494. *To construct an angle of* 12°; *of* 24°; *of* 6°.

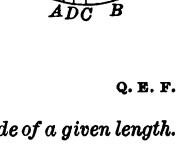
Let EF be an arc of a circle whose centre is O, and whose radius is OA.

1. To construct an angle of 24°.

Take arc AB equal to $\frac{1}{15}$ of the circumference. § 475

Draw OB.

$\angle AOB = \frac{1}{15}$ of 360° = 24°. § 288



2. To construct an angle of 12°.

Draw OC bisecting the $\angle AOB$.

$\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2}$ of 24° = 12°.

3. To construct an angle of 6°.

Draw OD bisecting the $\angle AOC$.

$\angle AOD = \frac{1}{2}\angle AOC = \frac{1}{2}$ of 12° = 6°. Q. E. F.

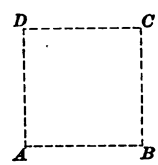
Ex. 495. *To construct an equilateral triangle with a side of a given length.*

Construction. Draw the straight line AB equal to the given length.

From A and B as centres with radii equal to AB describe arcs intersecting at C. Draw CA and CB.

The $\triangle ABC$ is the $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 496. *To construct a square with a side of a given length.*



Construction. Draw the straight line AB equal to the given length.

At A and B erect the $\perp$s AD and BC each equal to AB.

Draw CD.

The quadrilateral $ABCD$ is the square required.

Proof. $AD = BC = AB$. Const.

AD is $\parallel$ to BC. § 104

$\therefore ABCD$ is a $\square$, § 183

and AB is equal and parallel to DC.

$\therefore ABCD$ is equilateral.

Now $\angle A = \angle B = 90°$. Const.

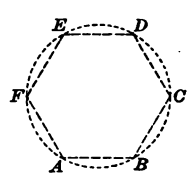
$\measuredangle$s C and D are the supplements of $\measuredangle$s A and B. § 115

$\therefore \angle C = \angle D = 90°$.

$\therefore ABCD$ is a square. § 168

Q. E. F.

Ex. 497. *To construct a regular hexagon with a side of a given length.*



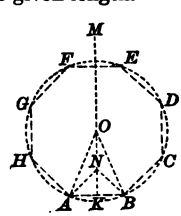
Construction. Describe the $\odot ABC$ with a radius equal to the given length.

From any point A of the circumference apply the given length as a chord six times.

$ABCDEF$ is the regular hexagon required. § 469

Q. E. F.

Ex. 498. *To construct a regular octagon with a side of a given length.*



Construction. Draw the straight line AB equal to the given length.

Draw the $\perp$ bisector KM of the line AB.

On KM take KN equal to AK, and draw AN, BN.

On NM take NO equal to AN, and draw AO, BO.

From O as a centre with a radius equal to AO describe a $\odot$.

Apply AB eight times as a chord to this $\odot$.

$ABCDEFGH$ is the regular octagon required.

Proof. $\angle NOA = \angle NAO$. § 145

$\angle ANK = \angle NAK = 45°$. § 145

But $\angle ANK = \angle NOA + \angle NAO$. § 137

$\therefore \angle NOA = \frac{1}{2}\angle ANK = 22\frac{1}{2}°$,

and $\angle AOB = 2\angle NOA = 45°$.

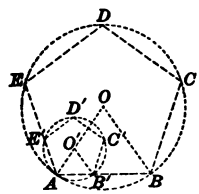
That is, $\angle AOB$ is equal to the angle at the centre of a regular octagon, being equal to $\frac{1}{8}$ of 360°, or 45°.

$\therefore ABCDEFGH$ is the regular octagon required. § 430

Q. E. F.

Ex. 499. *To construct a regular pentagon with a side of a given length.*

Construction. With any point O' as centre, with any convenient radius, describe a circle.



In this ⊙ inscribe a regular pentagon $A'B'C'D'E'$. § 473

On $A'B'$ or $A'B'$ produced take AB equal to the given length.

From B draw $BO \parallel$ to the radius $B'O'$ and let it meet $A'O'$ or $A'O'$ produced at O.

With O as centre and radius OB describe a ⊙, and apply the length AB as a chord five times.

The pentagon $ABCDE$ is the pentagon required.

Proof. $\angle BOA = \angle B'O'A$. § 112

$\therefore \angle BOA$ is equal to the angle at the centre of a regular pentagon.

$\therefore ABCDE$ is the regular pentagon required. Q. E. F.

Ex. 500. *To construct a regular decagon with a side of a given length.*

The construction and the proof is the same as in Ex. 499, with the exception that in the auxiliary ⊙ a regular decagon must be inscribed.

Q. E. F.

Ex. 501. *To construct a regular dodecagon with a side of a given length.*

The construction and the proof are the same as in Ex. 499, with the exception that in the auxiliary ⊙ a regular dodecagon must be inscribed.

Q. E. F.

Ex. 502. *To construct a regular pentedecagon with a side of a given length.*

The construction and the proof are the same as in Ex. 499, with the exception that in the auxiliary ⊙ a regular pentedecagon must be inscribed.

Q. E. F.

Ex. 503. *Find the area of a circle whose radius is* 12 *inches.*

12 inches = 1 foot.

Area ⊙ $= \pi R^2 = (3.1416 \times 1^2)$ sq. ft. $= 3.1416$ sq. ft. § 463

Ex. 504. *Find the circumference and the area of a circle whose diameter is* 8 *feet.*

Circumference $= 2\pi R = 3.1416 \times 8$ ft. $= 25.1328$ ft. § 458

Area $= \pi R^2 = (3.1416 \times 4^2)$ sq. ft. $= (3.1416 \times 16)$ sq. ft.

$= 50.2656$ sq. ft. § 463

Ex. 505. *A regular pentagon is inscribed in a circle whose radius is R. If the length of a side is a, find the apothem.*

$a = \frac{1}{2} R \sqrt{10 - 2\sqrt{5}}$. Ex. 463

$r^2 = R^2 - (\frac{1}{2} a)^2$ § 372

$= R^2 - (\frac{1}{2} R \sqrt{10 - 2\sqrt{5}})^2$

$= R^2 - \frac{1}{16} R^2 (10 - 2\sqrt{5})$

$= \frac{1}{16} R^2 (6 + 2\sqrt{5})$.

$\therefore r = \frac{1}{4} R \sqrt{6 + 2\sqrt{5}}$.

Ex. 506. *A regular polygon is inscribed in a circle whose radius is R. If the length of a side is a, show that the apothem is* $\frac{1}{4}\sqrt{4R^2 - a^2}$.

$r^2 = R^2 - (\frac{1}{2} a)^2 = R^2 - \frac{1}{4} a^2$ § 372

$\therefore 4r^2 = 4R^2 - a^2$.

$2r = \sqrt{4R^2 - a^2}$.

$r = \frac{1}{4}\sqrt{4R^2 - a^2}$.

Ex. 507. *Find the area of a regular decagon inscribed in a circle whose radius is* 16 *inches.*

$S = \frac{1}{2} rP$. § 459

$r = \frac{1}{4} R \sqrt{10 + 2\sqrt{5}} = \frac{1}{4} \times 16 \sqrt{10 + 2\sqrt{5}} = 4\sqrt{10 + 2\sqrt{5}}$. Ex. 443

$a = \frac{R(\sqrt{5} - 1)}{2} = \frac{16(\sqrt{5} - 1)}{2} = 8(\sqrt{5} - 1)$. Ex. 443

$\therefore P = 10a = 10 \times 8(\sqrt{5} - 1) = 80(\sqrt{5} - 1)$.

$\therefore S = \frac{1}{2} \times 4\sqrt{10 + 2\sqrt{5}} \times 80(\sqrt{5} - 1)$

$= 160(\sqrt{5} - 1)\sqrt{10 + 2\sqrt{5}}$

$= 160 \times (2.236 - 1)\sqrt{10 + 2 \times 2.23606}$

$= 160 \times 1.236 \times 3.804$

$= 752.28$. 752.28 sq. in. *Ans.*

Ex. 508. *Find the side of a regular dodecagon inscribed in a circle whose radius is* 20 *inches.*

$a = R\sqrt{2 - \sqrt{3}}$ Ex. 465

$= 20\sqrt{2 - \sqrt{3}}$

$= 20\sqrt{2 - 1.73205}$

$= 20 \times 0.517$

$= 10.34$. 10.34 in. *Ans.*

Ex. 509. *Find the perimeter of a regular pentagon inscribed in a circle whose radius is 25 feet.*

$a = \frac{1}{2} R\sqrt{10 - 2\sqrt{5}}$. Ex. 463

$P = 5a$

$= 5 \times \frac{1}{2} \times 25\sqrt{10 - 2\sqrt{5}}$

$= 62.5 \times \sqrt{10 - 2 \times 2.23606}$

$= 62.5 \times 2.351$

$= 146.94$. 146.94 sq. ft. *Ans.*

Ex. 510. *The length of each side of a park in the shape of a regular decagon is 100 yards. Find the area of the park.*

$S = \frac{1}{2} rP = \frac{1}{2} r \times 10a = 5ar$. § 459

Now $a = \frac{1}{2} R(\sqrt{5} - 1)$. Ex. 443

$\therefore R = \frac{2a}{\sqrt{5} - 1} = \frac{2a(\sqrt{5} + 1)}{5 - 1} = \frac{1}{2} a(\sqrt{5} + 1)$.

$r = \frac{1}{2}\sqrt{4R^2 - a^2}$ Ex. 506

$= \frac{1}{2}\sqrt{a^2(\sqrt{5} + 1)^2 - a^2}$

$= \frac{1}{2} a\sqrt{5 + 2\sqrt{5}}$.

$\therefore S = 5a \times \frac{1}{2} a\sqrt{5 + 2\sqrt{5}}$

$= \frac{5}{2} a^2 \times \sqrt{5 + 2 \times 2.23606}$

$= \frac{5}{2} \times 100^2 \times 3.077$

$= 76925$. 76,925 sq. yd. *Ans.*

Ex. 511. *Find the cost, at $2 per yard, of building a wall around a cemetery, in the shape of a regular hexagon, that contains 16,627.84 square yards.*

$S = \frac{1}{2} rP = \frac{1}{2} r \times 6a = 3ar$. § 459

$\therefore 16627.84 = 3ar$.

Now $a = R$, and $r = \frac{1}{2} R\sqrt{3}$. Ex. 442

$\therefore r = \frac{1}{2} a\sqrt{3}$.

$\therefore 3a \times \frac{1}{2} a\sqrt{3} = 16627.84$.

$\therefore a^2 = \frac{16627.84}{\frac{3}{2}\sqrt{3}} = \frac{16627.84}{\frac{3}{2} \times 1.732} = 6400$.

$\therefore a = \sqrt{6400} = 80$.

$P = 6a = 6 \times 80 = 480$.

$480 \times \$2 = \$960$. *Ans.*

Ex. 512. *The side of an inscribed regular polygon of n sides is* 16 *feet. Find the side of an inscribed regular polygon of* $2n$ *sides.*

Side of polygon of $2n$ sides $= \sqrt{R(2R - \sqrt{4R^2 - a^2})}$ § 478

$= \sqrt{R(2R - \sqrt{4R^2 - 16^2})}$

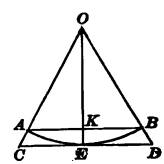
$= \sqrt{R(2R - \sqrt{4R^2 - 256})}$

$= \sqrt{2R(R - \sqrt{R^2 - 64})}.$

Ex. 513. *If the radius of a circle is R, and the side of an inscribed regular polygon is a, show that the side of the similar circumscribed regular polygon is* $\dfrac{2aR}{\sqrt{4R^2 - a^2}}$.

Let AB, the side of a regular inscribed polygon, be denoted by a, and the radius of the ⊙ by R.

Let CD be a side of the similar circumscribed regular polygon. § 441



To prove that $CD = \dfrac{2aR}{\sqrt{4R^2 - a^2}}$.

Proof. $OK = \sqrt{\overline{OA}^2 - \overline{AK}^2} = \sqrt{R^2 - \left(\frac{a}{2}\right)^2} = \frac{1}{2}\sqrt{4R^2 - a^2}.$ § 372

But $OK : OE = AB : CD.$ § 445

$\therefore \frac{1}{2}\sqrt{4R^2 - a^2} : R = a : CD.$

$\therefore CD = \dfrac{2aR}{\sqrt{4R^2 - a^2}}.$ § 327

Q. E. D.

Ex. 514. *What is the width of the circular ring between two concentric circumferences whose lengths are* 650 *feet and* 425 *feet?*

Let R be the radius of the larger ⊙, and R' the radius of the smaller.

Since $C = 2\pi R,$ § 458

$R = \dfrac{650}{2\pi}$ and $R' = \dfrac{425}{2\pi}.$

$\therefore R - R' = \dfrac{650}{2\pi} - \dfrac{425}{2\pi} = \dfrac{225}{2\pi} = \dfrac{112.5}{\pi} = 112.5 \times 0.31831 = 35.809875.$

35.8 ft. *Ans.*

Ex. 515. *Find the angle subtended at the centre by an arc* 5 *feet* 10 *inches long, if the radius of the circle is* 9 *feet* 4 *inches.*

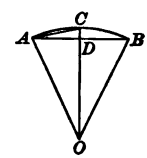
5 ft. 10 in. = 70 in.; 9 ft. 4 in. = 112 in.

$C = 2\pi R = 2 \times 3.1416 \times 112 = 703.7184.$ § 458

$703.7184 : 70 = 360° : \angle$ required. § 287

$\therefore \angle \text{required} = \dfrac{70 \times 360°}{703.7184} = 35.8°.$ *Ans.* § 327

Ex. 516. *The chord of a segment is* 10 *feet and the radius of the circle is* 16 *feet. Find the area of the segment.*



$AB = 10$ and $AO = 16 = BO$.

Area segment ACB = area sector $OACB$ − area $\triangle OAB$.

Inscribe in a ⊙ whose radius is unity a sector $O'A'C'B'$ similar to the sector $OACB$.

Then $A'B' : AB = O'B' : OB.$

$\therefore A'B' : 10 = 1 : 16.$

$\therefore A'B' = \frac{10}{16} = 0.625.$

The arc of the sector can be found approximately by a method similar to that used in § 480.

No. of Chords.			Sum of Chords.
C_1	$= 0.625.$		0.625
C_2	$= \sqrt{2 - \sqrt{4 - 0.625^2}}$	$= 0.3164877.$	0.6329754
C_4	$= \sqrt{2 - \sqrt{4 - 0.3164877^2}}$	$= 0.1587446.$	0.6349784
C_8	$= \sqrt{2 - \sqrt{4 - 0.1587446^2}}$	$= 0.0794350.$	0.6354800
C_{16}	$= \sqrt{2 - \sqrt{4 - 0.0794350^2}}$	$= 0.0397343.$	0.6357488
C_{32}	$= \sqrt{2 - \sqrt{4 - 0.0397343^2}}$	$= 0.0198683.$	0.6357856
C_{64}	$= \sqrt{2 - \sqrt{4 - 0.0198683^2}}$	$= 0.0099343.$	0.6357952

Therefore, the length of the arc AB is 0.6358, correct to four places of decimals.

$\therefore$ area sector $O'A'C'B' = \frac{1}{2} \times 0.6358 = 0.3179.$ § 462

$\therefore$ area sector $OACB = 16^2 \times 0.3179 = 81.3824.$ § 465

Area $\triangle OAB = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ Ex. 405

$= \sqrt{21 \times 11 \times 5 \times 5}$

$= 75.99.$

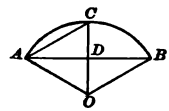
$\therefore$ area segment = 81.38 sq. ft. − 75.99 sq. ft. = 5.39 sq. ft.

Ex. 517. *Find the area of a sector, if the angle at the centre is* 20°, *and the radius of the circle is* 20 *inches.*

Area sector $= \frac{20}{360}$ of area $\odot$ § 462
$= \frac{20}{360} \times \pi R^2$ § 463
$= \frac{20}{360} \times 3.1416 \times 20^2$
$= 69.81.$ 69.81 sq. in. *Ans.*

Ex. 518. *The chord of half an arc is* 12 *feet, and the radius of the circle is* 18 *feet. Find the height of the segment subtended by the whole arc.*

$\overline{AC}^2 = 2\ OC \times CD.$ § 370
$12^2 = 2 \times 18 \times CD.$
$36\ CD = 144.$
$\therefore CD = 4.$ 4 ft. *Ans.*



Ex. 519. *Find the side of a square which is equivalent to a circle whose diameter is* 35 *feet.*

Area $\odot = \pi R^2 = \frac{1}{4}\pi D^2 = \frac{1}{4} \times 3.1416 \times 35^2 = 962.115.$ § 463
Side of square $= \sqrt{962.115} = 31.02.$ 31.02 ft. *Ans.*

Ex. 520. *The diameter of a circle is* 15 *feet. Find the diameter of a circle twice as large. Three times as large.*

Let S' and D' denote the area and diameter of the required $\odot$.

Then $S : S' = (\frac{1}{2}D)^2 : (\frac{1}{2}D')^2 = D^2 : D'^2.$ § 464

(1) $S : 2\ S = 15^2 : D'^2.$
$1 : 2 = 225 : D'^2.$
$\therefore D'^2 = 2 \times 225 = 450.$
$\therefore D' = \sqrt{450} = 21.21.$
21.21 ft. *Ans.*

(2) $S : 3\ S = 15^2 : D'^2.$
$1 : 3 = 225 : D'^2.$
$D'^2 = 3 \times 225 = 675.$
$\therefore D' = \sqrt{675} = 25.98.$
25.98 ft. *Ans.*

Ex. 521. *Find the radii of the concentric circumferences that divide a circle* 11 *inches in diameter into five equivalent parts.*

Let S' and R' denote the area and radius of the required $\odot$.

(1) $S : S' = R^2 : R'^2.$ § 464
$S : \frac{1}{5}\ S = (\frac{11}{2})^2 : R'^2.$
$5 : 1 = \frac{121}{4} : R'^2.$
$R'^2 = \frac{121}{20} = 6.05.$
$R' = \sqrt{6.05} = 2.46.$
2.46 in. *Ans.*

(2) $S : S' = R^2 : R'^2.$ § 464
$S : \frac{2}{5}\ S = (\frac{11}{2})^2 : R'^2.$
$5 : 2 = \frac{121}{4} : R'^2.$
$R'^2 = \frac{242}{20} = 12.1.$
$R' = \sqrt{12.1} = 3.48.$
3.48 in. *Ans.*

(3) $S : S' = R^2 : R'^2.$ § 464
$S : \frac{3}{5}\ S = (\frac{11}{2})^2 : R'^2.$
$5 : 3 = \frac{121}{4} : R'^2.$
$R'^2 = \frac{363}{20} = 18.15.$
$R' = \sqrt{18.15} = 4.26.$
4.26 in. *Ans.*

(4) $S : S' = R^2 : R'^2.$ § 464
$S : \frac{4}{5}\ S = (\frac{11}{2})^2 : R'^2.$
$5 : 4 = \frac{121}{4} : R'^2.$
$R'^2 = \frac{121}{5} = 24.2.$
$R' = \sqrt{24.2} = 4.29.$
4.92 in. *Ans.*

Ex. 522. *The perimeter of a regular hexagon is* 840 *feet, and that of a regular octagon is the same. By how many square feet is the octagon larger than the hexagon?*

Side of the hexagon $= \frac{1}{6}$ of 840 ft. $= 140$ ft.

Area of the hexagon $= 6$ times the area of an equilateral $\triangle$ of 140 ft. side.

$$\text{Area } \triangle = \frac{a^2\sqrt{3}}{4} = \frac{140^2 \times \sqrt{3}}{4} = 4900 \times 1.73205 = 8487.045. \quad \text{Ex. 404}$$

$\therefore$ area of hexagon $= 6 \times 8487.045$ sq. ft. $= 50{,}922.270$ sq. ft.

Side of the octagon $= \frac{1}{8}$ of 840 ft. $= 105$ ft.

$$a = R\sqrt{2 - \sqrt{2}}. \quad \text{Ex. 464}$$

$$\therefore R = \frac{a}{\sqrt{2 - \sqrt{2}}}.$$

$$r = \tfrac{1}{2}\sqrt{4R^2 - a^2} \quad \text{Ex. 506}$$

$$= \tfrac{1}{2}\sqrt{4\left(\frac{a}{\sqrt{2 - \sqrt{2}}}\right)^2 - a^2}$$

$$= \tfrac{1}{2}\sqrt{\frac{4a^2}{2 - \sqrt{2}} - a^2}$$

$$= \tfrac{1}{2}\sqrt{\frac{4a^2(2 + \sqrt{2})}{4 - 2} - a^2}$$

$$= \tfrac{1}{2}\sqrt{2a^2(2 + \sqrt{2}) - a^2}$$

$$= \tfrac{1}{2}\sqrt{3a^2 + 2a^2\sqrt{2}}$$

$$= \tfrac{1}{2}a\sqrt{3 + 2\sqrt{2}}.$$

$$S = \tfrac{1}{2}rP = \tfrac{1}{2}r \times 8a = 4ar. \quad \text{§ 459}$$

$$\therefore S = 4 \times 105 \times \tfrac{1}{2} \times 105 \times \sqrt{3 + 2\sqrt{2}}$$

$$= 2 \times 105^2 \times \sqrt{3 + 2 \times 1.41421}$$

$$= 2 \times 105^2 \times 2.414$$

$$= 53{,}228.700.$$

$53{,}228.700$ sq. ft. $- 50{,}922.270$ sq. ft. $= 2306.430$ sq. ft. *Ans.*

Ex. 523. *The diameter of a bicycle wheel is* 28 *inches. How many revolutions does the wheel make in going* 10 *miles?*

28 in. $= 2\frac{1}{3}$ ft.; 10 mi. $= 10 \times 5280$ ft. $= 52{,}800$ ft.

$$C = \pi D = 3.1416 \times 2\tfrac{1}{3} \text{ ft.} = 7.3304 \text{ ft.} \quad \text{§ 458}$$

Number of revolutions $= 52800 \div 7.3304 = 7202 +$.

Ex. 524. *Find the diameter of a carriage wheel that makes 264 revolutions in going half a mile.*

$\frac{1}{2}$ mi. $= \frac{1}{2}$ of 5280 ft. $= 2640$ ft.

Circumference of wheel $= \frac{2640}{264}$ ft. $= 10$ ft.

Diameter of wheel $= \frac{10}{3.1416} = 10 \times 0.31831 = 3.1831.$

3.1831 ft. *Ans.*

Ex. 525. *The sides of three octagons are 6 feet, 7 feet, 8 feet, respectively. Find the side of a regular octagon equivalent to the sum of the three given octagons.*

Let x be the side of the regular octagon required.

Two regular polygons of the same number of sides are to each other as the squares of their sides. § 446

$\therefore x^2 = 6^2 + 7^2 + 8^2.$ Ex. 381

$\therefore x = \sqrt{6^2 + 7^2 + 8^2} = \sqrt{36 + 49 + 64} = \sqrt{149} = 12.206.$

12.206 ft. *Ans.*

Ex. 526. *A circular pond 100 yards in diameter is surrounded by a walk 10 feet wide. Find the area of the walk.*

The radius of the pond is $\frac{1}{2}$ of 100 yd. $= 50$ yd. $= 150$ ft.

Area of the pond is $\pi R^2 = \pi \times 150^2 = \pi \times 22{,}500.$ § 463

Area of pond and walk is $\pi R^2 = \pi \times 160^2 = \pi \times 25{,}600.$

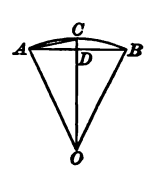
Area of walk is

$\pi \times 25{,}600 - \pi \times 22{,}500 = \pi \times 3100 = 3.1416 \times 3100 = 9738.96.$

9738.96 sq. ft. *Ans.*

Ex. 527. *The span (chord) of a bridge in the form of a circular arc is 120 feet, and the highest point of the arch is 15 feet above the piers. Find the radius of the arc.*

Let chord AB be 120 ft., and CD be 15 ft.



$\overline{AC}^2 = 2\,OC \times CD.$ § 370

But $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$ § 371

$= 60^2 + 15^2$

$= 3825.$

$OC = R - 15.$

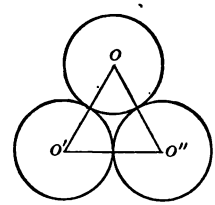
$\therefore 3825 = 2\,(R - 15) \times 15.$

$2\,(R - 15) = 255.$

$R - 15 = 127.5.$

$R = 142.5.$ 142.5 ft. *Ans.*

Ex. 528. *Three equal circles are described each tangent to the other two. If the common radius is R, find the area contained between the circles.*



The $\triangle OO'O''$ is equilateral, each side being $2R$, and the area is $R^2\sqrt{3}$. Ex. 404

The area of the sector cut out of each $\odot$ by the $\triangle$ is $\frac{1}{6}$ of the $\odot$, or $\frac{1}{6}\pi R^2$. § 462

$\therefore$ the area of the three sectors is $\frac{1}{2}\pi R^2$.

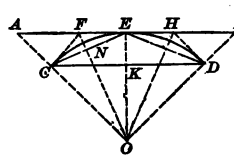
But the area contained between the Ⓢ is the area of the $\triangle$ diminished by the area of the three sectors; that is,

Area contained between the circles $= R^2\sqrt{3} - \frac{1}{2}\pi R^2 = \frac{1}{2}R^2(2\sqrt{3} - \pi)$.

Ex. 529. *Given p, P, the perimeters of regular polygons of n sides inscribed in and circumscribed about a given circle. Find p', P', the perimeters of regular polygons of 2n sides inscribed in and circumscribed about the given circle.*

Let AB be one side of the circumscribed polygon, and let CD, $\parallel$ to AB, be one side of the similar inscribed polygon.

To construct inscribed and circumscribed regular polygons of double the number of sides, and to compute their perimeters p', P' in terms of the perimeters p, P of the given polygons.



Construction. Draw $OE \perp$ to AB, and draw AO, BO, CE, DE.

Draw $CF \perp$ to AO, and $DH \perp$ to BO.

Then FH is a side of the circumscribed polygon required, and CE and DE sides of the inscribed polygon required.

Proof. C and D lie in AO and BO, respectively. § 441

$\therefore$ arc CE = arc DE. § 245

$\therefore$ $CE = DE$. § 241

$\therefore$ CE and DE are sides of the required inscribed polygon. § 442

CF and DH are tangent to the $\odot$. § 253

$\therefore$ FH is a side of the required circumscribed polygon. § 440

Computation. $$\frac{P}{p} = \frac{OA}{OE}.$$ § 447

Now the rt. $\triangle$ OFC and OFE are equal. § 151

For $FO = FO$ and $OC = OE$. § 217

$\therefore$ FO bisects the $\angle COE$. § 128

$$\therefore \frac{OA}{OE} = \frac{AF}{EF}.$$ § 348

$$\therefore \frac{P}{p} = \frac{AF}{EF}. \qquad \text{Ax. 1}$$

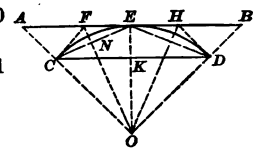
$$\therefore \frac{P+p}{p} = \frac{AF+EF}{EF}. \qquad \text{§ 332}$$

$$\therefore \frac{P+p}{2p} = \frac{AE}{2EF} = \frac{AE}{FH}.$$

But AE is contained as often in P as FH is in P'.

$$\therefore \frac{AE}{FH} = \frac{P}{P'}. \qquad \text{§ 340}$$

$$\therefore \frac{P+p}{2p} = \frac{P}{P'}. \qquad \text{Ax. 1}$$



Whence $$P' = \frac{2pP}{P+p}.$$

Also, the rt. $\triangle CKE$ and ENF are similar. § 356

For $$\angle ECK = \angle FEN. \qquad \text{§ 110}$$

$$\therefore \frac{CK}{CE} = \frac{EN}{EF}. \qquad \text{§ 351}$$

But $$\frac{CK}{CE} = \frac{p}{p'}, \qquad \text{§ 340}$$

and $$\frac{EN}{EF} = \frac{p'}{P'}. \qquad \text{§ 340}$$

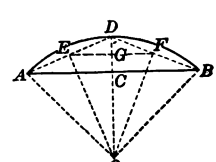
$$\therefore \frac{p}{p'} = \frac{p'}{P'}. \qquad \text{Ax. 1}$$

Whence $$p' = \sqrt{p \times P'}. \qquad \text{§ 327}$$

$$\therefore p' = \sqrt{\frac{2p^2P}{P+p}} = p\sqrt{\frac{2P}{P+p}}.$$

Ex. 530. *Given the radius R and the apothem r of an inscribed regular polygon of n sides. Find the radius R' and the apothem r' of an isoperimetrical regular polygon of $2n$ sides.*

Let AB be one side of the given regular polygon.



To construct an isoperimetrical regular polygon of double the number of sides, and to compute its radius R' and apothem r' in terms of the radius R and apothem r of the given polygon.

Construction. Draw $OD \perp$ to AB and draw AD, BD, OA, OB.

Draw $OE \perp$ to AD, and $OF \perp$ to BD. Draw EF.

Then EF is a side of the polygon required.

Proof. E and F are the middle points of AD and BD. § 245

$\therefore EF = \frac{1}{2}AB.$ § 189

$\therefore OE = R'$ and $OG = r'.$

Computation. Now $DG = GC.$ § 188

$$\therefore OG = \frac{OD + OC}{2}.$$

That is, $$r' = \frac{R + r}{2}.$$

Again $\overline{OE}^2 = OD \times OG.$ § 367

$$\therefore OE = \sqrt{OD \times OG}$$

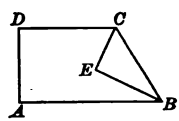
$$= \sqrt{OD \times \frac{OD + OC}{2}}.$$

$$\therefore R' = \sqrt{\frac{R(R + r)}{2}}.$$

MISCELLANEOUS EXERCISES.

Ex. 531. *If two adjacent angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.*

In the quadrilateral $ABCD$, let $\measuredangle A$ and D be rt. $\measuredangle$, and let BE bisect the $\angle ABC$ and CE bisect the $\angle BCD$.



To prove that BE is $\perp$ to CE.

Proof. $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\measuredangle$. § 205

But $\angle A + \angle D = 2$ rt. $\measuredangle$. Hyp.

$\therefore \angle B + \angle C = 2$ rt. $\measuredangle$. Ax. 3

$\therefore \frac{1}{2}\angle B + \frac{1}{2}\angle C = 1$ rt. $\angle$. Ax. 7

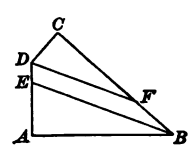
$\therefore \angle EBC + \angle ECB = 1$ rt. $\angle$.

$\therefore \angle BEC$ is a rt. $\angle$. § 131

$\therefore BE$ is $\perp$ to CE. Q. E. D.

Ex. 532. *If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.*

In the quadrilateral $ABCD$, let $\measuredangle A$ and C be rt. $\measuredangle$, and let BE bisect the $\angle ABC$ and DF bisect the $\angle CDA$.



To prove that BE and FD are $\parallel$.

Proof. $\angle A + \angle ABC + \angle C + \angle CDA = 4$ rt. $\measuredangle$. § 205

But $\angle A + \angle C = 2$ rt. $\measuredangle$. Hyp.

$\therefore \angle ABC + \angle CDA = 2$ rt. $\measuredangle$. Ax. 3

$\therefore \frac{1}{2}\angle ABC + \frac{1}{2}\angle CDA = 1$ rt. $\angle$. Ax. 7

$\therefore \angle EBA + \angle ADF = 1$ rt $\angle$.

But $\angle EBA + \angle AEB = 1$ rt. $\angle$. § 135

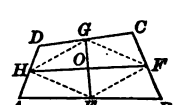
$\therefore \angle AEB = \angle ADF$. § 84

$\therefore BE$ and FD are $\parallel$. § 114

Q. E. D.

Ex. 533. *The two lines that join the middle points of the opposite sides of a quadrilateral bisect each other.*

Let $ABCD$ be a quadrilateral, and EG and FH be lines joining the middle points of the opposite sides.



To prove that EG and FH bisect each other.

Proof. Draw EF, FG, GH, HE.

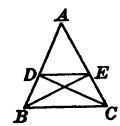
Then $EFGH$ is a $\square$ with EG and FH for diagonals. Ex. 74

$\therefore EG$ and FH bisect each other. § 184

Q. E. D.

Ex. 534. *The line that joins the feet of the perpendiculars dropped from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.*

Let ABC be an isosceles $\triangle$ with CD and $BE \perp$ to the equal sides AB and AC, respectively.



To prove that DE is $\parallel$ to BC.

Proof. The rt. $\triangleq$ CBD and BCE are equal. § 141

For $BC = BC$, Iden.

and $\angle CBD = \angle BCE$. § 145

$\therefore BD = CE$. § 128

$\therefore AB - BD = AC - CE$, or $AD = AE$. Ax. 3

That is, the $\triangle ADE$ is isosceles.

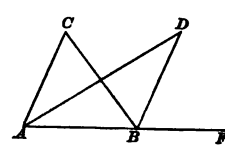
Now the $\angle A$ is common to the $\triangleq$ ADE and ABC.

$\therefore \angle ADE = \angle ABC$, and hence DE is $\parallel$ to BC. § 114

Q. E. D.

Ex. 535. *If AD bisects the angle A of a triangle ABC, and BD bisects the exterior angle CBF, then angle ADB equals one half angle ACB.*

Let AD bisect the $\angle A$ of the $\triangle ABC$, and let BD bisect the exterior $\angle CBF$.



To prove that $\angle D = \frac{1}{2}\angle C$.

Proof. $\angle CBF = \angle CAB + \angle C$. § 137

$\therefore \angle DBF = \angle DAB + \frac{1}{2}\angle C$. Ax. 7

But $\angle DBF = \angle DAB + \angle D$. § 137

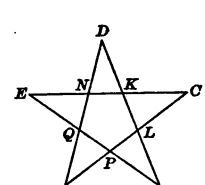
$\therefore \angle DAB + \frac{1}{2}\angle C = \angle DAB + \angle D$. Ax. 1

$\therefore \frac{1}{2}\angle C = \angle D$. Ax. 3

Q. E. D.

Ex. 536. *The sum of the acute angles at the vertices of a pentagram (five-pointed star) is equal to two right angles.*

Let A, B, C, D, E be the acute $\measuredangle$ at the vertices of the given pentagram.



To prove that $A + B + C + D + E = 2$ rt. $\measuredangle$.

Proof. $\angle DNK + \angle CKL + \angle BLP + \angle APQ + \angle EQN = 4$ rt. $\measuredangle$. § 207

Also $\angle DKN + \angle CLK + \angle BPL + \angle AQP + \angle ENQ = 4$ rt. $\measuredangle$. § 207

Hence, sum of base $\measuredangle$ of the $\triangleq$ DNK, CKL, etc. $= 8$ rt. $\measuredangle$. Ax. 2

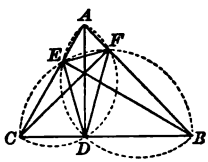
But sum of all the $\measuredangle$ of the $\triangleq$ DNK, CKL, etc. $= 10$ rt. $\measuredangle$. § 129

$\therefore A + B + C + D + E = 2$ rt. $\measuredangle$. Ax. 3

Q. E. D.

Ex. 537. *The altitudes AD, BE, CF of the triangle ABC bisect the angles of the triangle DEF.*

Let ED, EF, FD join the feet of the altitudes of the $\triangle ABC$.



To prove that $\angle EDA = \angle FDA$,

$\angle FEB = \angle DEB$,

and $\angle EFC = \angle DFC$.

Proof. On AB, BC, AC as diameters describe Ⓢ. These Ⓢ pass through D and E, E and F, D and F, respectively, since the $\measuredangle ADB$ and AEB, CEB and CFB, AFC and ADC are right $\measuredangle$. Ex. 170

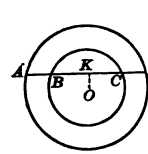
Now $\angle EDA = \angle EBA$ (each being measured by $\frac{1}{2}$ arc EA)
$= \angle ECF$ (each being measured by $\frac{1}{2}$ arc EF)
$= \angle FDA$ (each being measured by $\frac{1}{2}$ arc AF).

Also, $\angle FEB = \angle FCB$ (each being measured by $\frac{1}{2}$ arc FB)
$= \angle DAF$ (each being measured by $\frac{1}{2}$ arc DF)
$= \angle DEB$ (each being measured by $\frac{1}{2}$ arc DB).

Also, $\angle EFC = \angle EBC$ (each being measured by $\frac{1}{2}$ arc EC)
$= \angle EAD$ (each being measured by $\frac{1}{2}$ arc ED)
$= \angle DFC$ (each being measured by $\frac{1}{2}$ arc CD). Q. E. D.

Ex. 538. *The segments of any straight line intercepted between the circumferences of two concentric circles are equal.*

Let AD cut the outer ⊙ in A and D and the inner ⊙ in B and C.



To prove that $AB = DC$.

Proof. Draw OK from the centre $O \perp$ to AD.

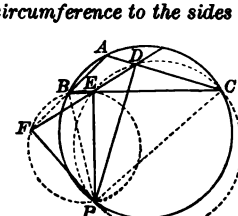
Then $KA = KD$ and $KB = KC$. § 245

$\therefore KA - KB = KD - KC$. Ax. 3

That is, $AB = DC$. Q. E. D.

Ex. 539. *If a circle is circumscribed about any triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides of the triangle lie in one straight line.*

Let P be any point in the circumference of the circle circumscribed about the $\triangle ABC$.



Let PD, PE, PF be $\perp$ to AC, BC, AB, respectively.

To prove that FED is a straight line.

Proof. Draw PB, PC; and on PB and PC as diameters describe Ⓢ.

Since $\measuredangle PEC$ and PDC are rt. $\measuredangle$, the $\odot$ on PC passes through E and D. Ex. 170

Since $\measuredangle PEB$ and PFB are rt. $\measuredangle$, the $\odot$ on PB passes through E and F. Ex. 170

Now $\angle BPE + \angle EPD + \angle DPC + \angle A = 2$ rt. $\measuredangle$; Ex. 97

and $\angle FPB + \angle BPE + \angle PBF + \angle PBE = 2$ rt. $\measuredangle$. Ex. 97

$\therefore \angle BPE + \angle EPD + \angle DPC + \angle A + \angle FPB + \angle BPE + \angle PBF + \angle PBE = 4$ rt. $\measuredangle$. Ax. 2

Again $\angle BPE + \angle EPD + \angle A + \angle ABE + \angle EBP = 3$ rt. $\measuredangle$. § 206

$\therefore \angle DPC + \angle BPF + \angle BPE + \angle PBF - \angle ABC = 1$ rt. $\angle$. Ax. 3

But $\angle ABC = \angle BFE + \angle BEF$; § 137

also $\angle BFE = \angle BPE$, and $\angle BEF = \angle BPF$. § 293

$\therefore \angle ABC = \angle BPE + \angle BPF$.

Substituting for $\angle ABC$ its equal, $\angle BPE + \angle BPF$,

$\angle DPC + \angle PBF = 1$ rt. $\angle$.

But $\angle DPC = \angle DEC$, and $\angle PBF = \angle PEF$. § 293

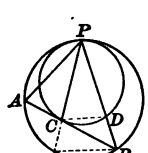
$\therefore \angle DEC + \angle PEF = 1$ rt. $\angle$.

$\therefore \angle DEC + \angle PEC + \angle PEF = 2$ rt. $\measuredangle$. Ax. 2

$\therefore FED$ is a straight line. § 90

Q. E. D.

Ex. 540. *Two circles are tangent internally at P, and a chord AB of the larger circle touches the smaller circle at C. Prove that PC bisects the angle APB.*



Let the ⑤ PCD and PAB be tangent internally at P. Let AB be a chord of the larger $\odot PAB$ tangent to the smaller $\odot PCD$ at C.

To prove that $\angle APC = \angle BPC$.

Proof. Produce PC meeting the larger $\odot$ at E, and draw CD and EB.

CD is $\parallel$ to EB. Ex. 262, § 345

Now $\angle APC = \angle ABE$, § 293

and $\angle ABE = \angle BCD$. § 110

Also $\angle BCD = \angle BPC$.

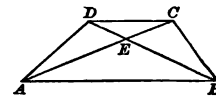
For $\frac{1}{2}$ arc CD measures $\angle BCD$, § 295

and $\frac{1}{2}$ arc CD measures $\angle BPC$. § 289

$\therefore \angle APC = \angle ABE$. Ax. 1

Q. E. D.

Ex. 541. *The diagonals of a trapezoid divide each other into segments which are proportional.*



Let $ABCD$ be a trapezoid whose diagonals AC and BD intersect at E.

To prove that $EA : EB = EC : ED$.

Proof. The $\triangle AEB$ and CED are similar. § 355

For $\angle AEB = \angle CED$, § 93

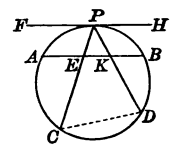
and $\angle EBA = \angle EDC$. § 110

$\therefore EA : EB = EC : ED$. § 351

Q. E. D.

Ex. 542. *If through a point P in the circumference of a circle two chords are drawn, the chords and the segments between P and a chord parallel to the tangent at P are reciprocally proportional.*

Let the chord AB be $\parallel$ to the tangent at P, and let PC and PD be chords cutting AB in E and K.



To prove that $PE : PK = PD : PC$.

Proof. Draw CD.

$\angle FPE = \angle D$. Ex. 101

But $\angle FPE = \angle PEK$. § 110

$\therefore \angle PEK = \angle D$. Ax. 1

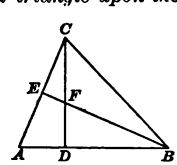
In like manner, $\angle PKE = \angle C$.

$\therefore$ the $\triangle PEK$ and PDC are similar. § 355

$\therefore PE : PK = PD : PC$. § 351

Q. E. D.

Ex. 543. *The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments reciprocally proportional.*



Let ABC be a $\triangle$, and let CD and BE be $\perp$, respectively, to AB and AC, and meet at F.

To prove that $FB : FC = FD : FE$.

Proof. The rt. $\triangle BDF$ and CEF are similar. § 356

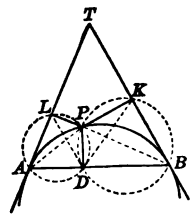
For $\angle BFD = \angle CFE$. § 93

$\therefore FB : FC = FD : FE$. § 351

Q. E. D.

Ex. 544. *The perpendicular from any point of a circumference upon a chord is the mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.*

Let AB be a chord, AT and BT tangents, PD drawn from any point P in the arc $\perp$ to AB. Let PL and PK be $\perp$ to AT and BT, respectively.



To prove that $PL : PD = PD : PK$.

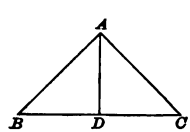
Proof. On BP and AP as diameters, describe Ⓢ. These Ⓢ will pass through K, D and L, D, respectively, since $\measuredangle PKB$, PDB, ALP, and ADP are rt. $\measuredangle$. Ex. 170

Draw LD and KD.

Now	$\angle PKD = \angle PBD.$	§ 293
	$\angle PBD = \angle LAP.$	
For	$\frac{1}{2}$ arc AP measures $\angle PBD$,	§ 289
and	$\frac{1}{2}$ arc AP measures $\angle LAP$.	§ 295
	$\angle LAP = \angle PDL.$	§ 293
	$\therefore \angle PKB = \angle PDL.$	Ax. 1
Again	$\angle PLD = \angle PAD.$	§ 293
	$\angle PAD = KBP.$	
For	$\frac{1}{2}$ arc PB measures $\angle PAD$,	§ 289
and	$\frac{1}{2}$ arc PB measures $\angle KBP$.	§ 295
	$\angle KBP = \angle PDK.$	§ 293
	$\therefore \angle PDK = \angle PLD.$	Ax. 1
	$\therefore$ the $\triangle LPD$ and DPK are similar.	§ 355
	$\therefore PL : PD = PD : PK.$	§ 351
		Q. E. D.

Ex. 545. *In an isosceles right triangle either leg is the mean proportional between the hypotenuse and the perpendicular upon it from the vertex of the right angle.*

Let BAC be an isosceles $\triangle$ with A a rt. $\angle$. Let AD be perpendicular to BC.



To prove that $BC : AB = AB : AD$.

Proof. $BC : AB = AB : BD.$ § 367

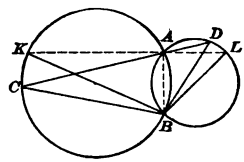
But $\angle B = \angle DAB = 45°.$

$\therefore BD = AD.$ § 147

$\therefore BC : AB = AB : AD.$ Q. E. D.

Ex. 546. *If two circles intersect in the points A and B, and through A any secant CAD is drawn limited by the circumferences at C and D, the straight lines BC, BD are to each other as the diameters of the circles.*

Let the two ⑤ intersect in the points A and B, and CAD be drawn, cutting the circumferences in C and D. Let BC and BD be drawn, also the diameters BK and BL.



To prove that $BC : BD = BK : BL$.

Proof. Draw AK and AL.

$\angle KAB$ and $\angle LAB$ are rt. $\measuredangle$. § 290

$\therefore AK$ and AL are in the same straight line. § 90

In the $\triangle CBD$ and KBL, $\angle C = \angle K$, and $\angle D = \angle L$. § 293

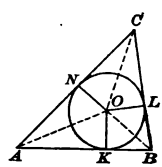
$\therefore$ the $\triangle CBD$ and KBL are similar. § 355

$\therefore BC : BD = BK : BL$. § 351

Q. E. D.

Ex. 547. *The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.*

Let ABC be a $\triangle$, and NLK be the inscribed ⊙ having the radius OK.



To prove that area $\triangle = \frac{1}{2}(AB + BC + CA) \times OK$.

Proof. The ⊥ OK, OL, ON are all equal. § 162

Area $\triangle AOB = AB \times \frac{1}{2} OK$,

area $\triangle BOC = BC \times \frac{1}{2} OL$,

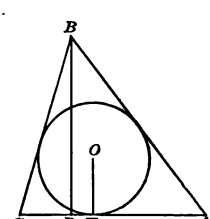
area $\triangle AOC = AC \times \frac{1}{2} ON$.

$\therefore$ Area $(\triangle AOB + \triangle BOC + \triangle AOC) = \frac{1}{2}(AB + BC + AC) \times OK$. Ax. 2

$\therefore$ area $\triangle ABC = \frac{1}{2}(AB + BC + AC) \times OK$. Q. E. D.

Ex. 548. *The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.*

Let ABC be a $\triangle$ whose perimeter is p. Let BD be ⊥ to CA, and OK be the radius of the inscribed ⊙.



To prove that $p : AC = BD : OK$.

Proof. area $\triangle = \frac{1}{2} p \times OK$. Ex. 547

Also, area $\triangle = \frac{1}{2} AC \times BD$. § 403

$\therefore \frac{1}{2} p \times OK = \frac{1}{2} AC \times BD$. Ax. 1

$\therefore p \times OK = AC \times BD$. Ax. 6

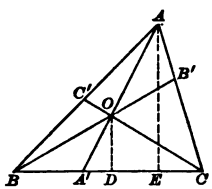
$\therefore p : AC = BD : OK$. § 329

Q. E. D.

Ex. 549. *If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then*

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

Let O be any point in the $\triangle ABC$. Let AA', CC', BB' be drawn through O, cutting the sides BC, AB, AC in A', C', B', respectively.



To prove that $\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$

Proof. Draw OD and $AE \perp$ to BC.

The rt. $\triangle ODA'$ and AEA' are similar. § 356

$$\therefore \frac{OD}{AE} = \frac{OA'}{AA'}. \qquad \text{§ 351}$$

$$\frac{\triangle OBC}{\triangle ABC} = \frac{OD}{AE}. \qquad \text{§ 405}$$

$$\therefore \frac{\triangle OBC}{\triangle ABC} = \frac{OA'}{AA'}. \qquad \text{Ax. 1}$$

In like manner, $\frac{\triangle OAC}{\triangle ABC} = \frac{OB'}{BB'}$, and $\frac{\triangle OAB}{\triangle ABC} = \frac{OC'}{CC'}$.

$$\therefore \frac{\triangle OBC + \triangle OAC + \triangle OAB}{\triangle ABC} = \frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'}. \qquad \text{Ax. 2}$$

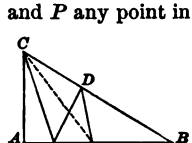
But $$\frac{\triangle OBC + \triangle OAC + \triangle OAB}{\triangle ABC} = \frac{\triangle ABC}{\triangle ABC} = 1.$$

$$\therefore \frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1. \qquad \text{Ax. 1}$$

Q. E. D.

Ex. 550. *ABC is a triangle, M the middle point of AB, P any point in AB between A and M. If MD is drawn parallel to PC, meeting BC at D, the triangle BPD is equivalent to half the triangle ABC.*

In the $\triangle ABC$, let M be the middle point of AB, and P any point in AM. Let MD be drawn $\parallel$ to PC. Draw PD.



To prove that $\triangle BPD \doteq \frac{1}{2} \triangle ABC$.

Proof. Draw MC.

Now $BM = \frac{1}{2} AB$. Hyp.

$\therefore \triangle BMC \doteq \frac{1}{2} \triangle ABC$. § 405

In $\triangle BMC$ and BPD, $\triangle BMD$ is common.

$\triangle MPD \doteq \triangle MDC$. § 404

$\therefore \triangle BMC \doteq BPD$. Ax. 2

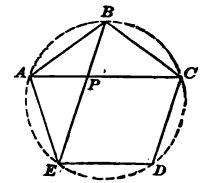
$\therefore \triangle BPD \doteq \frac{1}{2} \triangle ABC$. Ax. 1

Q. E. D.

Ex. 551. *Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.*

Let BE and AC be diagonals of a regular pentagon and intersect at P.

To prove that $AC : PC = PC : AP$.



Proof. Circumscribe a ⊙ about the pentagon.

The ≙ ABC and APB are similar. § 355

For $\angle BAC = \angle BAP$, Iden.

and $\angle ACB = \angle ABP$. § 293

$\therefore AC : AB = AB : AP$. § 351

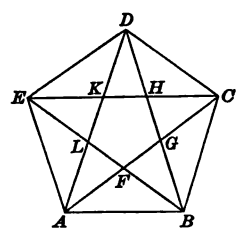
But $PC = AB$. Ex. 466

Substitute PC for its equal AB.

Then $AC : PC = PC : AP$. Q. E. D.

Ex. 552. *If all the diagonals of a regular pentagon are drawn, another regular pentagon is thereby formed.*

Let all the diagonals of the regular pentagon $ABCDE$ be drawn, forming the pentagon $FGHKL$.



To prove that $FGHKL$ is a regular pentagon.

Proof. $\triangle ABC = \triangle BCD = \triangle CDE$

$= \triangle DEA = \triangle EAB$. § 143

For $AB = BC = CD = DE = EA$, § 429

and $\angle ABC = \angle BCD = \angle CDE = \angle DEA$

$= \angle EAB$. § 429

$\therefore AC = BD = CE = DA = EB$. § 128

Now $AG = BH = CK = DL = EF = FC = GD = HE = KA = LB$. Ex. 466

$\therefore FG = GH = HK = KL = LF$.

Also, $FB = GB = GC = HC = HD = KD = KE = LE = LA = FA$. Ax. 3

$\therefore$ ≙ BFG, CGH, DHK, EKL, ALF are equal isosceles ≙. § 150

$\therefore \angle BFG = \angle CGH = \angle DHK = \angle EKL = \angle ALF$. § 128

$\therefore \angle LFG = \angle FGH = \angle GHK = \angle HKL = \angle KLF$. § 85

$\therefore$ pentagon $FGHKL$, being equilateral and equiangular, is regular. § 429

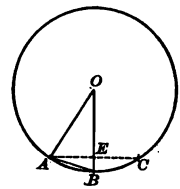
Q. E. D.

Ex. 553. *The area of an inscribed regular dodecagon is equal to three times the square of the radius.*

Let AB be the side of a regular dodecagon whose area is P inscribed in a circle whose centre is O and radius R.

To prove that $P = 3R^2$.

Proof. Draw AC, a side of a regular inscribed hexagon.



Area of $\triangle AOB = OB \times \frac{1}{2} AE$

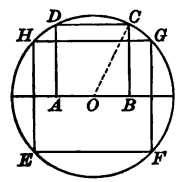
$= R \times \frac{1}{2} R = \frac{1}{2} R^2$. § 403

But the $\triangle AOB$ is $\frac{1}{12}$ of the dodecagon.

Hence, $P = 12 \times \frac{1}{2} R^2 = 3R^2$. Q. E. D.

Ex. 554. *The area of a square inscribed in a semicircle is equal to two fifths the area of the square inscribed in the circle.*

Let $ABCD$ be a square inscribed in the semicircle, and $EFGH$ be a square inscribed in the ⊙ whose centre is O.



Now area of $ABCD = \overline{BC}^2$,

and area of $EFGH = \overline{EF}^2$. § 398

To prove that $\overline{BC}^2 : \overline{EF}^2 = 2 : 5$.

Proof. Draw OC.

$\overline{BC}^2 = \overline{OC}^2 - \overline{OB}^2$. § 372

$\therefore \overline{BC}^2 = \overline{OC}^2 - (\frac{1}{2}\overline{BC})^2$.

$\therefore \overline{BC}^2 = \overline{OC}^2 - \frac{1}{4}\overline{BC}^2$.

$\therefore \overline{BC}^2 = \frac{4}{5}\overline{OC}^2$.

$EF = OC\sqrt{2}$. Ex. 441

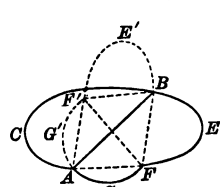
$\therefore \overline{EF}^2 = 2\ \overline{OC}^2$.

$\therefore \overline{BC}^2 : \overline{EF}^2 = \frac{4}{5}\overline{OC}^2 : 2\ \overline{OC}^2 = \frac{4}{5} : 2 = 2 : 5$. Q. E. D.

Ex. 555. *The area of a circle is greater than the area of any polygon of equal perimeter.*

Let $ACBFG$ be the maximum figure formed with a given perimeter.

To prove that $ACBFG$ is a circle.



Proof. From any point A in the perimeter draw AB so as to divide the perimeter into two equal parts.

AB divides the figure $ACBFG$ into two equivalent parts.

For if the area of one of the parts, as AFB, is greater than that of the other part, ACB, then, if the part AFB is revolved upon AB as an axis into the position $AF'B$, the figure $AG'F'E'B$ would be greater than the figure ACB, but would have the same perimeter.

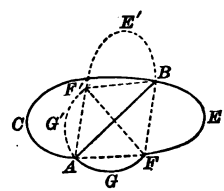
Hence, the figure ACB would not be the maximum, which is contrary to the hypothesis.

Since ACB is the maximum figure, $AG'F'E'B$ is also the maximum figure, for it has the same perimeter and the same area as ACB.

Now $AG'F'E'B$ is symmetrical with respect to the line AB, since by the revolution about AB, every line $FF' \perp$ to AB, terminated by the perimeter, is bisected by AB. § 211

Since F and F' are two symmetrical points in the perimeter, the $\measuredangle AFB$ and $AF'B$ are equal.

Now the $\measuredangle AFB$ and $AF'B$ are rt. $\measuredangle$.



For, if they were not rt. $\measuredangle$, the areas of the $\measuredangle AFB$ and $AF'B$ could be increased without changing the lengths of the chords AF, FB, AF', $F'B$ (§ 484), and then (the segments AGF, FEB, $AG'F'$, $F'E'B'$, still standing on these chords), the whole figure would have its area increased without changing the length of its perimeter.

Therefore, $AG'F'E'B$ would not be the maximum.

Therefore, the $\measuredangle F$ and F' are rt. $\measuredangle$.

But F is any point in AFB.

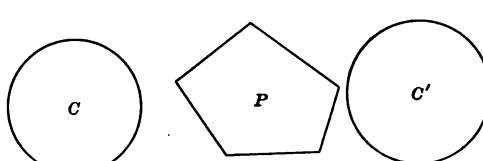
Therefore, AFB is a semicircumference. Ex. 170

Therefore, any half the figure being a semicircle, the whole figure $ACBFG$ is a circle. Q. E. D.

Ex. 556. *The circumference of a circle is less than the perimeter of any polygon of equal area.*

Let C be a circle, and P be any polygon having the same area as C.

To prove that the perimeter of C is less than the perimeter of P.



Proof. Let C' be a circle having the same perimeter as P.

Then $P < C'$, or $C < C'$. Ex. 555

Now, of two Ⓢ, that which has the less perimeter has the less area.

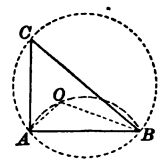
Therefore, the perimeter of $C <$ the perimeter of C'.

Therefore, the perimeter of $C <$ the perimeter of P. Q. E. D.

Ex. 557. *Find the locus of the centre of the circle inscribed in a triangle that has a given base and a given angle at the vertex.*

Let ABC be a triangle with the given base AB, and the vertical $\angle C$ equal to the given $\angle$.

Let O be the centre of the $\odot$ inscribed in the $\triangle ABC$.



To find the locus of O.

Analysis. Suppose the construction effected and O a point in the locus required.

The locus of the vertex C is the arc of the segment on AB as a chord in which the given $\angle$ may be inscribed.

The centre of the inscribed $\odot$ is the intersection of the bisectors AO and BO of the $\measuredangle$ CAB and CBA.

Now $\angle C = 180° - (\angle A + \angle B)$, § 131

and $\angle O = 180° - (\frac{1}{2}\angle A + \frac{1}{2}\angle B)$

$= 90° + \frac{1}{2}\angle C$.

Therefore, $\angle O$ is constant, and O lies on the arc of the segment on AB as a chord in which may be inscribed an $\angle$ equal to $90°$ + half the given $\angle$.

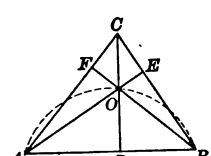
Construction. On AB as a chord, describe a segment of a $\odot$ in which may be inscribed an angle equal to $90°$ + half the given $\angle$.

The arc of this segment is the locus required. Q. E. F.

Ex. 558. *Find the locus of the intersection of the altitudes of a triangle that has a given base and a given angle at the vertex.*

Let $\triangle ABC$ be one of the $\measuredangle$ that have the base AB and the vertical $\angle C$ equal to the given $\angle$.

Let the altitudes AE, BF, CD intersect at O.



To find the locus of O.

In the quadrilateral $OECF$, the sum of the $\measuredangle$ = 4 rt. $\measuredangle$. § 206

But $\angle OEC = \angle OFC = 1$ rt. $\angle$. Hyp.

$\therefore \angle FOE = 180° - \angle FCE$.

But $\angle FOE = \angle AOB$. § 93

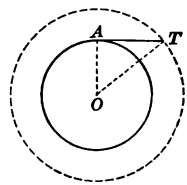
$\therefore \angle AOB = 180° - \angle FCE$. Ax. 1

That is, the $\angle AOB$ is constant and equal to the supplement of the given vertical $\angle$ of the $\triangle$.

Hence, the locus of the point O is the arc of the segment of a $\odot$ constructed on AB as a chord in which may be inscribed an angle equal to the supplement of the given vertical $\angle$ of the $\triangle$. Q. E. F.

Ex. 559. *Find the locus of the extremity of a tangent to a given circle, if the length of the tangent is equal to a given line.*

Let AT be tangent to the ⊙ whose centre is O at any point A of the circumference.



Take AT equal to the given length.

To find the locus of T.

Draw OA and OT.

OA is $\perp$ to AT. § 254

$\therefore \overline{OT}^2 = \overline{AT}^2 + \overline{OA}^2$. § 371

$\therefore OT = \sqrt{\overline{AT}^2 + \overline{OA}^2}$.

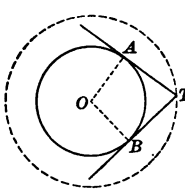
That is, the length OT is constant and equal to $\sqrt{\overline{AT}^2 + \overline{OA}^2}$.

Hence, the locus of the point T is the circumference of a circle concentric with the given ⊙ having for radius the distance from the centre of the given ⊙ to the extremity of a tangent of the given length.

Q. E. F.

Ex. 560. *Find the locus of a point, tangents drawn from which to a given circle form a given angle.*

Let O be the centre of the given ⊙ and let TA and TB be tangents drawn from the same point T, and let $\angle ATB$ be equal to the given $\angle$.



To find the locus of O.

Draw the radii OA and OB.

Then the sum of the $\measuredangle$s of the quadrilateral $AOBT = 4$ rt. $\measuredangle$s. § 206

But $\measuredangle A$ and B are rt. $\measuredangle$s. § 254

$\therefore \angle O$ is the supplement of $\angle T$.

Therefore, the angle at the centre formed by the radii to the points of contact of the tangents from any point of the locus is constant and equal to the supplement of the given $\angle$.

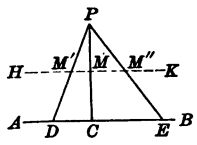
Hence, the locus of the point T is the circumference of a circle concentric with the given ⊙ having for radius the distance from the centre of the given ⊙ to the point of intersection of any two tangents that include the given $\angle$.

Q. E. F.

Ex. 561. *Find the locus of the middle point of a line drawn from a given point to a given straight line.*

Let P be the given point, AB the given line. Let PC be the $\perp$ from P to the line AB, and let PD and PE be any other lines drawn from P to AB.

To find the locus of the middle points of these lines.



Draw HK the $\perp$ bisector of PC.

HK is the locus required.

Proof. HK is $\parallel AB$. § 104

Since HK bisects PC, it bisects all other lines from P to AB. § 188

Hence, the required locus is a line $\parallel$ to the given line at a distance from the given line equal to half the distance of the given point from the given line and on the same side of the given line as is the given point.

Q. E. F.

Ex. 562. *Find the locus of the vertex of a triangle that has a given base and a given altitude.*

The locus of the vertex is evidently a line $\parallel$ to the given base at a distance from the given base equal to the given altitude. Q. E. F.

Ex. 563. *Find the locus of a point the sum of whose distances from two given parallel lines is equal to a given length.*

Let d equal the distance between the given parallel lines, and let l equal the given length.

If $l > d$, the locus consists of two lines parallel to the given parallel lines lying without the given lines, each at a distance $\frac{1}{2}(l - d)$ from the nearer line.

If $l = d$, every point between the lines satisfies the condition.

If $l < d$, there is no locus. Q. E. F.

Ex. 564. *Find the locus of a point the difference of whose distances from two given parallel lines is equal to a given length.*

Let d equal the distance between the given parallel lines, and let l equal the given length.

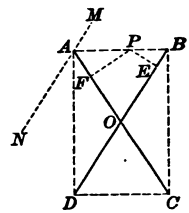
If $l < d$, the locus consists of two lines between the given parallel lines and parallel to them each at a distance $\frac{1}{2}(d - l)$ from the nearer line.

If $l = d$, every point in either of the given parallel lines satisfies the condition, and every point outside the given parallel lines satisfies the given condition.

If $l > d$, there is no locus. Q. E. F.

Ex. 565. *Find the locus of a point the sum of whose distances from two given intersecting lines is equal to a given length.*

Let AC and BD be the given lines intersecting at O, and let l be the given length.



To find the locus of a point the sum of whose distances from AC and $BD = l$.

Draw $MN \parallel BD$ at a distance equal to l. Let MN cut AC at A.

Take OB, OC, and OD equal to OA.

Draw AB, BC, CD, DA.

Then $ABCD$ is a $\square$. Ex. 68

Also $ABCD$ is a rectangle. Ex. 70

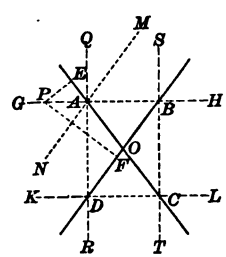
The sides of the rectangle $ABCD$ is the required locus.

Proof. The sum of the $\perp$s from any point P in any one of the sides of the rectangle is equal to the given length l. Ex. 60

Q. E. F.

Ex. 566. *Find the locus of a point the difference of whose distances from two given intersecting lines is equal to a given length.*

Let AC and BD be the given lines intersecting at O, and let l be the given length.



To find the locus of a point the difference of whose distances from AC and $BD = l$.

Construct the rectangle $ABCD$ as in Ex. 565, and prolong each of its sides both ways.

The prolongations of the sides of the rectangle is the locus required.

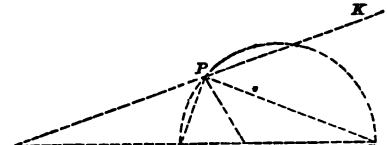
Proof. The difference of the $\perp$s PF and PE, drawn from any point P in AG, is equal to l.

For $VF = l$, and $PE = PV$, since rt. $\triangle PEA$ = rt. $\triangle PVA$. § 141

Q. E. F.

Ex. 567. *Find the locus of a point whose distances from two given points are in the given ratio $m : n$.*

Let A and B be the two given points, and $m : n$ the given ratio. Let P be taken so that $PA : PB = m : n$.



To find the locus of P.

Analysis. Suppose the construction effected, and P a point in the locus required.

Divide AB internally at C and externally at D, in the ratio $m : n$. §§ 348, 349

Then $PA : PB = m : n = CA : CB = AD : BD$.

Now PC bisects the $\angle APB$, and PD bisects the $\angle BPK$.

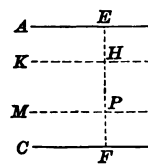
$\therefore CPD$ is a rt. $\angle$. Ex. 29

$\therefore P$ is in the circumference on CD as a diameter. Ex. 170

$\therefore$ the required locus is this circumference.

Construction. Divide AB harmonically at C and D in the ratio $m : n$ (§ 350), and upon CD as a diameter describe a circumference. Q. E. F.

Ex. 568. *Find the locus of a point whose distances from two given parallel lines are in the given ratio $m : n$.*



Let AB and CD be the given parallel lines.

Draw any line $EF \perp$ to AB and CD.

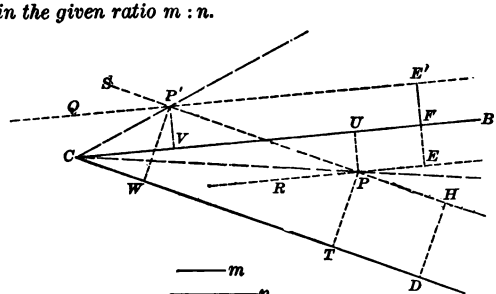
Divide EF at H so that $EH : HF = m : n$. § 385

Through H draw $KL \parallel$ to AB.

Again divide FE at P so that $FP : PE = m : n$.

The $\parallel$ lines KL and MN are evidently the required locus. Q. E. F.

Ex. 569. *Find the locus of a point whose distances from two given intersecting lines are in the given ratio $m : n$.*



Let CA and CB be the two given lines, and $m : n$ the given ratio.

To find the locus of a point whose distances from CB and CA are in the ratio $m : n$.

Construction. From any point on CA, as D, draw DH equal to n, and $\perp$ to CA. Through H draw $HS \parallel$ to AC. From any point on CB, as F, draw FE equal to m, and $\perp$ to CB, and also FE' equal to m, and $\perp$ to CB. Through E draw $ER \parallel$ to BC, intersecting HS in P, and through E' draw $E'Q \parallel$ to BC, intersecting HS in P'.

Then the locus required consists of the two straight lines CP and CP'.

Proof. Since $PT (\perp$ to $CA) = HD = n$, and $PU (\perp$ to $CB) = EF = n$, the point P is a point of the locus required.

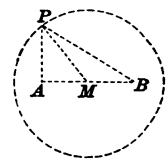
Any other point on CP will also satisfy the required condition; for $\perp$s from this point on CB and CA will, by similarity of $\triangle$s, be as $m : n$.

Therefore CP is one branch of the required locus.

In like manner, CP' may be shown to be another branch. Q. E. F.

Ex. 570. *Find the locus of a point the sum of the squares of whose distances from two given points is constant.*

Let A and B be the given points, and let P be any point of the required locus.



To find the locus of P such that $\overline{PA}^2 + \overline{PB}^2$ is constant.

Draw AB and bisect AB at M. Draw PA, PB, PM.

Let $\overline{PA}^2 + \overline{PB}^2 = k^2$, a constant quantity.

Now $\overline{PA}^2 + \overline{PB}^2 = 2\,\overline{AM}^2 + 2\,\overline{PM}^2$. § 377

That is, $k^2 = 2\,\overline{AM}^2 + 2\,\overline{PM}^2$.

Whence, $PM = \sqrt{\frac{1}{2}(k^2 - 2\,\overline{AM}^2)}$, a constant.

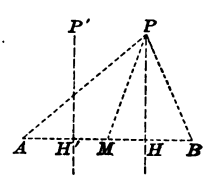
Hence, the required locus is the circumference of a circle whose centre is the middle point of the line that joins the two given points. Q. E. F.

Discussion. If k^2 is known, and $k^2 < 2\,\overline{AM}^2$; that is, if $k < AM\sqrt{2}$, PM is an imaginary number and there is no locus.

If $k^2 = 2\,\overline{AM}^2$, that is, if $k = AM\sqrt{2}$, $PM = 0$, and the circumference of the circle is reduced to the point M.

Ex. 571. *Find the locus of a point the difference of the squares of whose distances from two given points is constant.*

Let A and B be the given points, and let P be any point of the required locus.



To find the locus of P such that $\overline{PA}^2 - \overline{PB}^2$ is constant.

Draw AB and bisect AB at M. Draw PA, PB, PM, and draw $PH \perp$ to AB.

Let $\overline{PA}^2 - \overline{PB}^2 = k^2$, a constant quantity.

Now $\overline{PA}^2 - \overline{PB}^2 = 2\,AB \times MH$. § 377

That is, $k^2 = 2\,AB \times MH$.

Whence $MH = \dfrac{k^2}{2\,AB}$, a constant.

Hence, the locus is two straight lines perpendicular to the line that joins the two given points at equal distances from the middle point of that line. Q. E. F.

Discussion. If k^2 is known, the distance MH is the third proportional to the lengths $2\,AB$ and k. The other part of the locus is the line $\perp$ to AB erected at the distance MH on the other side of M.

Ex. 572. *Find the locus of the vertex of a triangle that has a given base and the other two sides in the given ratio $m : n$.*

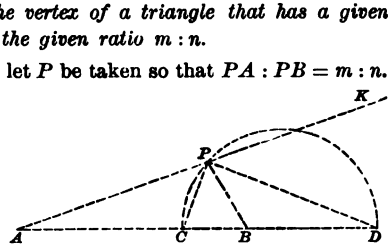
Let AB be the given base, and let P be taken so that $PA : PB = m : n$.

To find the locus of P.

Divide AB harmonically at C and D in the ratio $m : n$. § 350

Divide BA harmonically at C' and D' in the ratio $m : n$. § 350

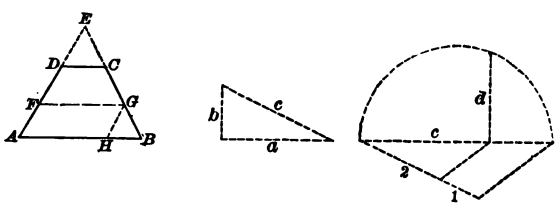
The circumferences of circles on CD and $C'D'$ as diameters are the locus required.



Ex. 567

Q. E. F.

Ex. 573. *To divide a given trapezoid into two equivalent parts by a line parallel to the bases.*



Let $ABCD$ be the given trapezoid.

To divide the trapezoid $ABCD$ into two equivalent parts by a line ∥ to AB.

Construction. Draw a rt. △ with the leg a equal to AB, and the leg $b = DC$.

Find d so that $d^2 : c^2 = 1 : 2$. § 427

On AB take AH equal to d, and draw HG ∥ to AD, and FG ∥ to AB.

FG is the line required.

Proof. Produce AD and BC until they meet at E.

$FG = AH = d$. § 180

Since DC, FG, and AB are ∥, ▲ ABE, FGE, and DCE are similar.

$$\therefore \triangle FGE : \triangle DCE = \overline{FG}^2 : \overline{DC}^2, \quad (1)$$

and $$\triangle ABE : \triangle DCE = \overline{AB}^2 : \overline{DC}^2. \quad (2)$$ § 411

From (1), by division,

$$\triangle FGE - \triangle DCE : \triangle DCE = \overline{FG}^2 - \overline{DC}^2 : \overline{DC}^2.$$ § 333

But $$\overline{FG}^2 = \tfrac{1}{2} c^2 = \tfrac{1}{2} (a^2 + b^2) = \tfrac{1}{2} (\overline{AB}^2 + \overline{DC}^2).$$ Const.

Substituting for $\overline{FG}^2$ its equal $\frac{1}{2} (\overline{AB}^2 + \overline{DC}^2)$,

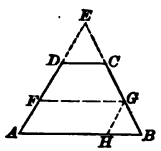
$$\triangle FGE - \triangle DCE : \triangle DCE = \tfrac{1}{2} (\overline{AB}^2 - \overline{DC}^2) : \overline{DC}^2. \quad (3)$$

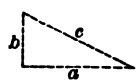
From (2), by division,

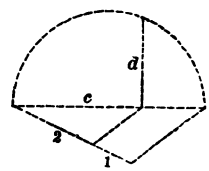
$$\triangle ABE - \triangle DCE : \triangle DCE = \overline{AB}^2 - \overline{DC}^2 : \overline{DC}^2. \quad (4)$$

From (3) and (4),

$$\triangle FGE - \triangle DCE : \triangle ABE - \triangle DCE = \tfrac{1}{2}(\overline{AB}^2 - \overline{DC}^2) : \overline{AB}^2 - \overline{DC}^2.$$







But $\triangle FGE - \triangle DCE =$ the trapezoid $FGCD$,

and $\triangle ABE - \triangle DCE =$ the trapezoid $ABCD$.

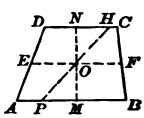
$\therefore FGCD : ABCD = \tfrac{1}{2} : 1.$

$\therefore$ the trapezoid $FGCD \simeq \tfrac{1}{2}$ the trapezoid $ABCD$. Q. E. F.

Ex. 574. *To divide a given trapezoid into two equivalent parts by a line through a given point in one of the bases.*

Let $ABCD$ be the given trapezoid and P the given point.

To divide the trapezoid $ABCD$ into two equivalent parts by a line through P.



1. Suppose that $AP > AB - DC$.

Construction. Draw the median EF and draw the altitude NM through O, the middle point of EF.

Through P and O draw PH, meeting DC at H.

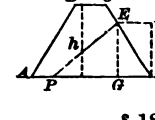
Then PH is the line required.

Proof. The trapezoids $APHD$ and $PBCH$ have the equal medians EO and OF, and the same altitude NM.

$\therefore$ the trapezoid $APHD \simeq$ the trapezoid $PBCH$. § 408

2. Suppose that $AP < AB - DC$.

Construction. Draw the altitude h. Find BF, the fourth proportional to $\tfrac{1}{2}BP$, $\tfrac{1}{2}(AB + DC)$, and h.



Draw $BF \perp$ to AB at B, $FE \parallel$ to BA, and $EG \parallel$ to FB, and draw PE.

Then PE is the line required.

Proof. $EG = FB.$ § 180

$\therefore \tfrac{1}{2}BP : \tfrac{1}{2}(AB + DC) = h : EG.$

$\therefore \tfrac{1}{2}(BP \times EG) = \tfrac{1}{2}(AB + DC)\,h.$ § 327

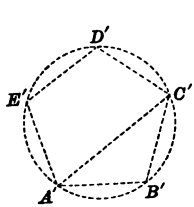
The area of the $\triangle PBE = \tfrac{1}{2}(BP \times EG).$ § 403

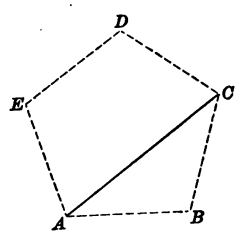
The area of the trapezoid $ABCD = \tfrac{1}{2}(AB + DC)\,h.$ § 407

$\therefore \triangle PBE \simeq \tfrac{1}{2}$ trapezoid $ABCD.$ Q. E. F.

Ex. 575. *To construct a regular pentagon, given one of the diagonals.*

Let AC be the given diagonal.





To construct a regular pentagon.

Construction. Inscribe a regular pentagon $A'B'C'D'E'$ in any $\odot A'C'E'$. § 473

Draw $A'C'$.

On AC, homologous to $A'C'$, construct $\triangle ABC$ similar to $\triangle A'B'C'$.

On AC, homologous to $A'C'$, construct the quadrilateral $ACDE$ similar to the quadrilateral $A'C'D'E'$. § 391

$ABCDE$ is the regular pentagon required. Q. E. F.

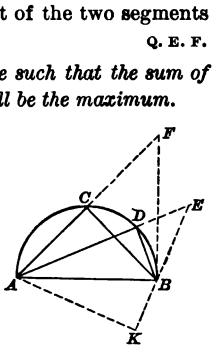
Ex. 576. *To divide a straight line into two segments such that their product shall be the maximum.*

Since the maximum of isoperimetric polygons of the same number of sides is a regular polygon (§ 489), the maximum quadrilateral whose perimeter is equal to twice the given line is a regular quadrilateral, that is, a square; and each side of the square is equal to half the given line.

Therefore, bisect the given line, and the product of the two segments is the maximum. Q. E. F.

Ex. 577. *To find a point in a semicircumference such that the sum of its distances from the extremities of the diameter shall be the maximum.*

Let ACB be a semicircle, C the middle point of the arc ACB, and D any other point.



To prove that $AC + BC > AD + BD$.

Proof. Produce AC to F, making CF equal to AC. Draw BF, and produce AD to E, making DE equal to DB. Draw EB, and produce it to meet AK drawn through $A \perp$ to EB.

Since $CA = CB = CF$, B is on the semicircumference on AF as a diameter.

$\therefore ABF$ is a rt. $\angle$. § 290

$\angle ACB = \angle F + \angle CBF = 2\angle F$. §§ 137, 145

$\therefore \angle F = \frac{1}{2} ACB = \frac{1}{2} ADB$. § 293

$\angle ADB = \angle E + \angle DBE = 2\angle E$. §§ 137, 145

$\therefore \angle E = \frac{1}{2} ADB = \angle F$. Ax. 1

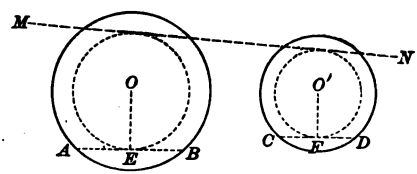
$\therefore$ the rt. $\triangle AFB$ and AEK are similar. § 356

$\therefore AB : AK = AF : AE$. § 351

But $AB > AK$ (§ 97). $\therefore AF > AE$.

That is, $AC + BC > AD + BD$. Q. E. F.

Ex. 578. *To draw a common secant to two given circles exterior to each other such that the intercepted chords shall have the given lengths a, b.*



Let O and O' be the centres of two Ⓢ exterior to each other.

To draw a common secant so that the intercepted chords shall have the given lengths a and b.

Construction. In the ⊙ whose centre is O, draw a chord AB equal to the given length a, and in the ⊙ whose centre is O', draw a chord CD equal to the given length b. Draw $OE \perp$ to AB and $O'F \perp$ to CD.

With centre O and radius OE, describe a circumference, and with centre O' and radius $O'F$, describe a circumference.

Now any chord in the given ⊙ with centre O tangent to the concentric ⊙ is equal to a, and any chord in the given ⊙ with centre O' tangent to the concentric ⊙ is equal to b. § 249

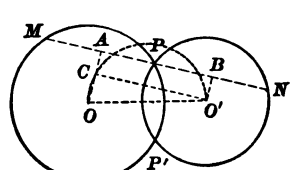
Draw MN a common tangent to the auxiliary Ⓢ. Ex. 245

MN is the common secant required, as is obvious from the construction. Q. E. F.

Discussion. Since the auxiliary Ⓢ are exterior to each other, two common external and two common internal tangents may be drawn, each of which satisfies the given condition.

Ex. 579. *To draw through one of the points of intersection of two intersecting circles a common secant which shall have a given length.*

Let O and O' be the centres of the two given Ⓢ and P one of the points of intersection of the two Ⓢ.



To draw through P a common secant which shall have a given length m.

Construction. Draw the line of centres OO', and on OO' as diameter describe a semicircle.

With O' as centre and radius equal to $\frac{1}{2}m$, describe an arc, cutting the semicircle at C.

Draw CO.

Through P draw the common secant $MN \parallel$ to CO.

Then MN is the common secant required.

Proof. Produce OC to meet MN at A, and draw $O'B \perp$ to MN.

OA is $\perp$ to MN. § 107

$\therefore MA = AP$,

and $PB = BN$. § 245

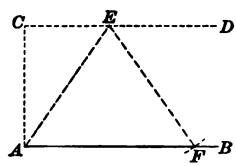
$\therefore AB = \frac{1}{2} MN$.

But $AB = CO' = \frac{1}{2} m$. § 180

$\therefore MN = m$. Q. E. F.

Ex. 580. *To construct an isosceles triangle, given the altitude and one of the equal base angles.*

Construction. At any point A of a straight line AB draw $AC \perp$ to AB and equal to the given altitude.



Draw $CD \parallel$ to AB.

At A draw AE cutting CD at E, making $\angle EAB$ equal to the given base $\angle$.

With E as a centre and radius EA, describe an arc, cutting AB at F.

Then $\triangle AEF$ is the isosceles $\triangle$ required.

Proof. The altitude is equal to the given altitude. Const.

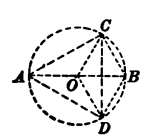
$EF = EA$. Const.

$\therefore \angle EAF = \angle EFA =$ given $\angle$. § 145

Q. E. F.

Ex. 581. *To construct an equilateral triangle, given the altitude.*

Construction. Draw a straight line AO equal to $\frac{2}{3}$ of the given altitude.



With centre O and radius OA, describe a $\odot$.

Draw the diameter AOB.

With centre B and radius OA, cut the $\odot$ at C and D.

Draw AC, AD, CD.

Then $\triangle ACD$ is the equilateral $\triangle$ required.

Proof. Draw OC, OD, BC, BD.

The $\triangle OBC$ is equilateral by construction.

$\therefore \angle BOC = 60°$. § 146

$\therefore \angle COD = 120°$.

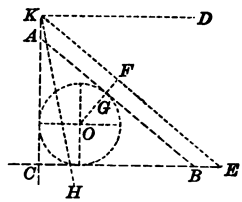
Now $\angle AOC = \angle AOD = 120°$. § 85

$\therefore$ arc $AC =$ arc $AD =$ arc DC. § 236

$\therefore AC = AD = DC$. § 241

Q. E. F.

Ex. 582. *To construct a right triangle, given the radius of the inscribed circle and the difference of the acute angles.*



Construction. Describe a ⊙ with the given radius. Draw two diameters ⊥ to each other, and draw a tangent to the ⊙ through one end of each. These tangents KC and EC form a rt. ∠. At any point K of the tangent CK draw KH, making $\angle CKH$ equal to the given difference of the acute ∡s. Draw $KD \parallel$ to CE and bisect the $\angle HKD$. Let the bisector cut the tangent CE at E. Draw $OF \perp$ to KE, cutting the circumference at G. Through G draw a tangent meeting CK at A, and CE at B.

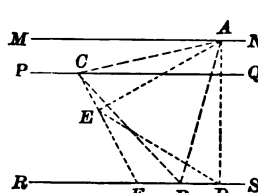
Then $\triangle ABC$ is the rt. △ required.

Proof.

	CK is ⊥ to KD.	§ 107
	$\therefore \angle HKD = 90° -$ given difference.	
	$\therefore \angle EKD = \frac{1}{2}(90° -$ given difference$)$.	
	$\therefore \angle EKC = \frac{1}{2}(90° +$ given difference$)$.	
Now	$\angle EKD = \angle KEC$,	§ 110
and	$\angle KEC = \angle ABC$.	§ 112
	$\therefore \angle ABC = \angle EKD$.	Ax. 1
Again	$\angle EKC = \angle BAC$.	§ 112
	$\therefore \angle BAC - \angle ABC =$ the given difference.	Q. E. F.

Ex. 583. *To construct an equilateral triangle so that its vertices shall lie in three given parallel lines.*

Let MN, PQ, and RS be any three ∥ lines.



To construct an equilateral △ so that its vertices shall lie in MN, PQ, and RS.

Construction. Draw $AD \perp$ to the ∥ lines MN, PQ, and RS.

On AD as one side construct the equilateral $\triangle ADE$.

Draw $CF \perp$ to AE at E and terminated by the ∥ lines PQ and RS.

Draw AC.

Construct the $\angle CAB$ equal to 60°, and draw BC.

Then the $\triangle ABC$ is the equilateral △ required.

Proof. Since the $\angle CAB = 60°$, and the $\angle EAD = 60°$, Const.

$\angle CAE = \angle BAD$.	Ax. 3
$EA = AD$.	
$\therefore$ rt. $\triangle CEA =$ rt. $\triangle BAD$.	§ 142

$\therefore AC = AB.$ § 128

Since the $\angle CAB = 60°$, Const.

the $\angle ACB = 60°$. § 145

$\therefore$ the $\triangle ABC$ is equiangular.

$\therefore$ the $\triangle ABC$ is equilateral. § 146

Q. E. F.

Ex. 584. *To draw a line from a given point to a given straight line which shall be to the perpendicular from the given point as $m : n$.*

Let AB be the given line, and PC the $\perp$ from the given point to AB.

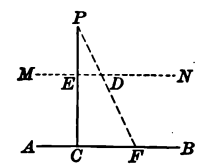
To draw a line from P to AB which shall be to PC as $m : n$.

Construction. On PC take PE equal to n, and through E draw $MN \parallel$ to AB.

With centre P and radius m, cut MN at D.

Draw PD, and produce it to meet AB at F.

Then PF is the line required.



Proof. $PF : PD = PC : PE.$ § 343

$\therefore PF : PC = PD : PE.$ § 330

But $PD = m$, and $PE = n$. Const.

$\therefore PF : PC = m : n.$ Q. E. F.

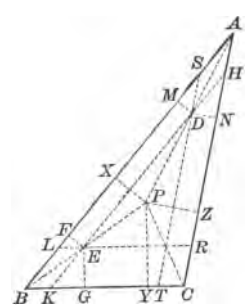
Ex. 585. *To find a point within a given triangle such that the perpendiculars from the point to the three sides shall be as the numbers m, n, p.*

Let ABC be the given $\triangle$.

To find a point within the $\triangle ABC$ such that the $\perp$s on AB, BC, AC shall be as $m : n : p$.

Construction. Draw $HK \parallel$ to AB at a distance equal to m from AB, $LR \parallel$ to BC at a distance equal to n from BC, and $ST \parallel$ to AC at a distance equal to p from AC. Let HK intersect LR at E, and ST at D. Let AD produced meet BE produced at P.

Then P is the required point.



Proof. From E draw EF and $EG \perp$ to AB and BC, respectively.

From D draw DM and $DN \perp$ to AB and AC, respectively.

From P draw PX, PY, $PZ \perp$ to AB, BC, and AC, respectively.

$\triangle BPX$ and BEF are similar, and $\triangle BPY$ and BEG are similar. § 356

$\therefore PX : EF = PB : EB = PY : EG.$ § 351

$\therefore PX : PY = EF : EG = m : n.$ § 330

Similarly, $PX : PZ = m : p.$

$\therefore PX : PY : PZ = m : n : p.$ Q. E. F.

Ex. 586. *To draw a straight line equidistant from three given points.*

Let A, B, C be the three given points.

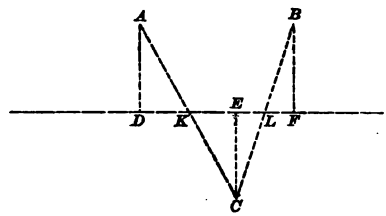
To draw a line equidistant from A, B, and C.

Construction. Draw AC and BC.

Bisect AC and BC in K and L, respectively.

Through K and L draw DF.

Then DF is the required line.



Proof. Draw AD, BF, $CE \perp$ to DF.

The rt. $\triangle ADK$ and CEK are equal. § 141

For $AK = CK$, Const.

and $\angle AKD = \angle CKE$. § 93

$\therefore AD = CE$. § 128

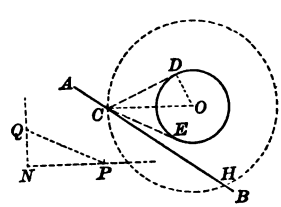
In like manner, $CE = BF$. Q. E. F.

Discussion. If the three given points are in a straight line, the required line is any line $\parallel$ to the line in which the points lie.

Ex. 587. *To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.*

Let DE be the given $\odot$, m the given length, and AB the given line.

To find a point on AB from which a tangent to the $\odot$ shall be of the given length m.



Analysis. Suppose C to be the required point on AB, and CD the required tangent. Join the centre O with C and D.

Then $\overline{OC}^2 = \overline{OD}^2 + \overline{CD}^2$ § 371

$= \overline{OD}^2 + m^2$.

Construction. Draw $NP \perp$ to NQ, and take NP equal to m and NQ equal to OD. Draw QP. With O as centre and QP as radius describe a $\odot$ cutting AB in C. From C draw the tangent CD.

Then CD is the tangent required.

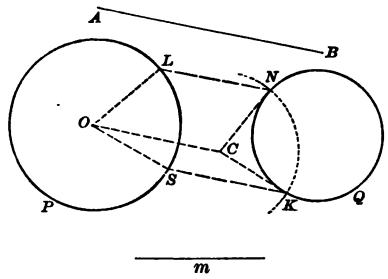
Proof. $CD = \sqrt{\overline{CO}^2 - \overline{OD}^2} = \sqrt{\overline{QP}^2 - \overline{NQ}^2} = NP = m$. Q. E. F.

Discussion. From C another tangent CE may be drawn which will fulfil the required conditions. Two other tangents from H may also be

drawn. So that there are in general *four* tangents possible, two from each of the points in which the constructed ⊙ cuts AB. It is evident that if the line AB is at the distance OC from the centre, the two points C and H coincide, and but two tangents may be drawn; and if the line AB is at a greater distance from the centre than OC, the constructed ⊙ does not meet AB and there is no solution.

Ex. 588. *To inscribe a straight line of a given length between two given circumferences and parallel to a given straight line.*

Let LP and NQ be the two given ⑤, AB the given straight line, and m the given length.



To draw a line ∥ to AB, so that the given length m shall be intercepted between the two circumferences.

Analysis. Suppose LN the line required. Draw OL, and draw OC equal to m, and ∥ to LN. Draw CN.

Since $LN = OC$, and LN is ∥ to OC, $OCNL$ is a □. § 183

$\therefore CN = OL$ and CN is ∥ to OL.

Construction. Draw OC equal to m, and ∥ to AB.

With C as centre and radius equal to the radius of the ⊙ LP, describe an arc cutting the ⊙ NQ in N.

Draw OL ∥ to CN, and cutting the ⊙ LP in L.

Draw LN.

Then LN is the line required.

Proof.

CN is equal and ∥ to OL.	Const.
$\therefore LN$ is equal and ∥ to OC.	§ 183
But OC is equal to m and ∥ to AB.	Const.
$\therefore LN$ is equal to m, and is ∥ to AB.	Q. E. F.

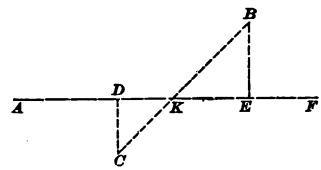
Discussion. Since, in general, the arc NK cuts the ⊙ NQ in another point K, a second line SK will fulfil the required conditions.

If the arc NK touches the ⊙ NQ, LN and SK will coincide, and only one construction is possible.

If the arc NK does not touch the ⊙ NQ, the problem is impossible.

Ex. 589. *To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio.*

Let A be the given point through which the straight line is to be drawn, B and C the other given points, and $m : n$ the given ratio.



To draw AF so that the $\perp$s from B and C shall be in the ratio $m : n$.

Analysis. Suppose AF the line required, and BE and $CD \perp$ to AF. Draw BC, and let BC cut AF in K.

The rt. $\triangle BKE$ and CKD are similar, § 356

For $\angle BKE = \angle CKD$. § 93

$\therefore BK : CK = BE : CD = m : n$. § 351

Construction. Draw BC. Divide BC at K in the ratio $m : n$. § 385

Through A and K draw AF.

Then AF is the required line.

Proof. The rt. $\triangle BKE$ and CKD are similar. § 356

$\therefore BE : CD = BK : CK = m : n$. § 351

Q. E. F.

Ex. 590. *To construct a square equivalent to the sum of a given triangle and a given parallelogram.*

Construction. Construct a square equivalent to the $\triangle$, by finding the mean proportional between the base and half the altitude of the $\triangle$. § 422.

Construct a square equivalent to the $\square$, by finding the mean proportional between the base and altitude of the $\square$. § 421

Then construct a square equivalent to the sum of the two squares. § 417

Q. E. F.

Ex. 591. *To construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.*

Construction. Construct a square equivalent to the given $\triangle$. § 422

Construct a square equivalent to the given pentagon. § 423

Construct a square equivalent to the sum of these squares. § 417

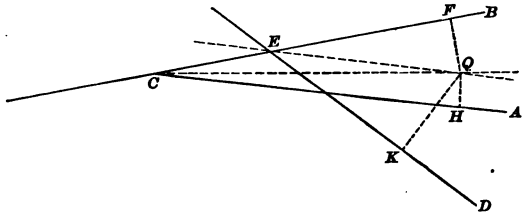
Then construct a rectangle equivalent to this last square and having the difference of its base and altitude the given line. § 425

Q. E. F.

Ex. 592. *To construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.*

This is a special case of § 426. Q. E. F.

Ex. 593. *To find a point whose distances from three given straight lines shall be as the numbers m, n, p.*



Let CA, CB, ED be the three given lines.

To find a point Q such that its distances to the three lines in order shall be as m, n, p.

Construction. Construct CQ, the locus of a point whose distances from CA and CB are as $m : n$. Ex. 569

Construct EQ, the locus of a point whose distances from CB and ED are as $n : p$.

Let CQ and EQ intersect in Q.

Then Q is the point required.

Proof. Draw $QH \perp$ to CA, $QF \perp$ to CB, and $QK \perp$ to ED.

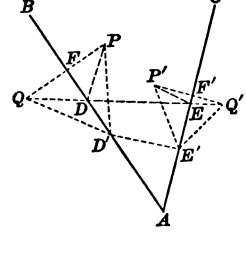
By construction, $QH : QF : QK = m : n : p$. Ex. 569

Q. E. F.

Ex. 594. *Given an angle and two points P and P′ between the sides of the angle. To find the shortest path from P to P′ that touches both sides of the angle.*

Let BAC be the given $\angle$, and P and P' the given points.

To find the shortest path from P to P' that touches AB and AC.



Construction. Draw $PF \perp$ to AB and produce it to Q, making FQ equal to PF.

Draw $P'F' \perp$ to AC and produce it to Q', making $F'Q'$ equal to $P'F'$.

Draw QQ' cutting AB at D and AC at E. Draw PD and $P'E$.

Then $PDEP'$ is the shortest path required.

Proof. Draw any other path $PD'E'P'$ from P to P' touching AB and AC.

Draw $D'Q$ and $E'Q$.

AB is the $\perp$ bisector of PQ and AC is the $\perp$ bisector of $P'Q'$. Const.

$\therefore PD = DQ$,

$P'E = EQ'$,

$PD' = D'Q$,

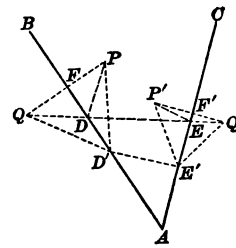
and $P'E' = E'Q'$. § 160

$\therefore PDEP' = QQ'$,

and $PD'E'P' = QD'E'Q'$. Ax. 2

But $QQ' < QD'E'Q'$. § 49

$\therefore PDEP' < PD'E'P'$. Q. E. F.

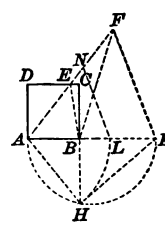


Ex. 595. *To construct a triangle, given its angles and its area.*

Let BD be a given square, and R and S given $\measuredangle$.

To construct a $\triangle$ having an area BD, and two of its $\measuredangle$ equal to R and S.

Construction. Draw AF, making $\angle FAB$ equal to $\angle R$. Let AF cut DC in E. Take EF equal to AE, and draw FB. Draw FK, making the $\angle AFK$ equal to $\angle S$, and cutting AB produced in K. On AK as a diameter describe a semicircle, cutting CB produced in H. Draw AH. Take AL on AK equal to AH. Through L draw $LN \parallel$ to KF.



Then $\triangle ANL$ is the $\triangle$ required.

Proof. $\triangle ANL$ has the required $\measuredangle$, since $\angle NAL = \angle R$ and $\angle ANL = \angle AFK$ (§ 112) $= \angle S$.

Draw EB. $\triangle AEB \doteq \frac{1}{4}$ square BD. § 403

Now, $\triangle AFB \doteq 2\,\triangle AEB$, since these $\measuredangle$ have the same altitude, and the base AF equal to $2\,AE$. Const.

$\therefore \triangle AFB \doteq$ square BD.

Again, $\overline{AH}^2 = \overline{AL}^2 = AB \times AK$. § 370

$\triangle AKF : \triangle ALN = \overline{AK}^2 : \overline{AL}^2$. § 411

$\therefore \overline{AK}^2 : AB \times AK = AK : AB = \triangle AFK : \triangle AFB$. § 405

That is, $\triangle AKF : \triangle ALN = \triangle AFK : \triangle AFB$.

$\therefore \triangle ALN \doteq \triangle AFB \doteq$ square BD. Q. E. F.

Ex. 596. *To transform a given triangle into a triangle similar to another given triangle.*

To transform the $\triangle ABC$ into a $\triangle$ similar to the given $\triangle PQR$.

Analysis. Let AFG be the required $\triangle$, constructed with the side AF in the line AB.

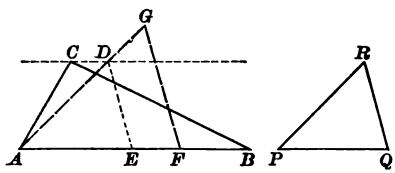
Draw $CD \parallel$ to AB, meeting AG in D, and draw $DE \parallel$ to GF.

Then the $\triangle AED$ and AFG are similar. § 354

$$\therefore \triangle AED : \triangle AFG = \overline{AE}^2 : \overline{AF}^2. \quad \text{§ 411}$$

Now $\triangle AFG \backsim \triangle ABC$. Hyp.

$$\therefore \triangle AED : \triangle ABC = \overline{AE}^2 : \overline{AF}^2.$$



Since the $\triangle AED$ and ABC have equal altitudes,

$$\triangle AED : \triangle ABC = AE : AB. \quad \text{§ 405}$$

$$\therefore \overline{AE}^2 : \overline{AF}^2 = AE : AB. \quad \text{Ax. 1}$$

$$\therefore \overline{AF}^2 = AB \times AE. \quad \text{§ 327}$$

That is, AF is the mean proportional between the two known lines AB and AE.

Construction. Place the $\triangle ABC$ and PQR so that their bases AB and PQ shall lie in the same straight line.

Draw $AG \parallel$ to PR, meeting at D the line through $C \parallel$ to AB.

Draw $DE \parallel$ to RQ, meeting AB at E.

Find AF, the mean proportional between AE and AB.

Draw $FG \parallel$ to QR, meeting AG at G.

Then the $\triangle AFG$ is the $\triangle$ required, as is obvious from the construction.

Q. E. F.

Ex. 597. *Given three points A, B, C. To find a fourth point P such that the areas of the triangles APB, APC, BPC shall be equal.*

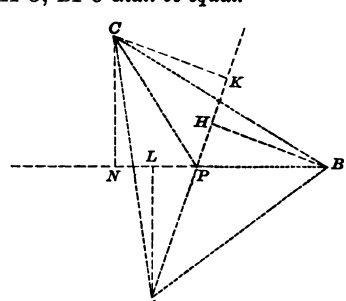
Let A, B, C be the three given points.

To find a fourth point P, such that $\triangle APB \backsim \triangle APC \backsim \triangle BPC$.

Analysis. Suppose the problem solved, and P the required fourth point.

The $\triangle BPC$ and APB are equivalent, and have a common base PB.

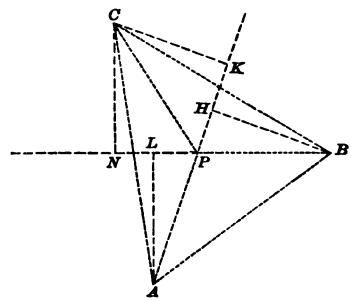
Hence, their altitudes CN and AL are equal. § 405



In like manner, since the △ APB and APC are equivalent, and have a common base AP, their altitudes BH and CK are equal.

But since $CN = AL$,

AC must be bisected by BP; § 142
and in like manner, since $CK = BH$,
BC must be bisected by AP.



Construction. Draw AB, BC, CA.
Draw BP and AK through the middle points of AC and BC, respectively, intersecting at P.

Then P is the fourth point required.

Proof. Since BC is bisected by AP,
Const.
the two rt. △ formed by the halves of BC and the ⊾ BH and CK are equal. § 141

$$\therefore BH = CK.$$

$$\therefore \triangle APB \fallingdotseq \triangle APC. \quad \text{§ 404}$$

In like manner, $\therefore \triangle APB \fallingdotseq \triangle BPC.$

$$\therefore \triangle APB \fallingdotseq \triangle APC \fallingdotseq \triangle BPC. \quad \text{Q. E. F.}$$

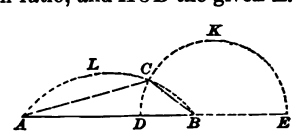
Ex. 598. *To construct a triangle, given its base, the ratio of the other sides, and the angle included by them.*

Let AB be the given base, $m : n$ the given ratio, and ACB the given ∠.

To construct a △ ABC so that

$$CA : CB = m : n,$$

and $\angle ACB =$ the given ∠.



Analysis. Since $CA : CB = m : n$, if AB is divided harmonically at D and E, in the given ratio, C must lie on the circumference described on DE as diameter. Ex. 567

Since $\angle ACB =$ the given ∠, C must lie on the arc ALB, which contains the given ∠. § 318

Hence, the point C is determined by the intersection of the arcs ALB and DKE.

Construction. Divide AB harmonically at D and E.

Upon DE as a diameter describe the semicircle DKE.

Upon AB describe the segment ALB containing the given ∠. § 318

Let the arc ALB cut the arc DKE at C.

Draw AC and CB.

Then ACB is the required △.

Proof. $\angle ACB$ is equal to the required ∠, and $CA : CB = m : n$. Ex. 567

Q. E. F.

Ex. 599. *To divide a given circle into n equivalent parts by concentric circumferences.*

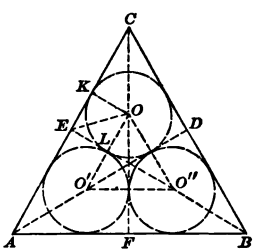
Solution. The method employed in Ex. 492 is applicable to the division of a ⊙ into any number of equal parts, the number of equal parts into which the radius is divided corresponding to the required number of parts of the ⊙. Q. E. F.

Ex. 600. *In a given equilateral triangle to inscribe three equal circles tangent to each other, each circle tangent to two sides of the triangle.*

Let ABC be an equilateral $\triangle$.

To draw three equal Ⓢ, each tangent to the other two and to two sides of the $\triangle$.

Analysis. Each of these Ⓢ is evidently inscribed in a quadrilateral formed by the sides and altitudes of the $\triangle$, as $ECDS$ (S, not shown in the figure, being the intersection of the altitudes).



The centre of the ⊙ must lie on CF, as CF is the locus of all points equidistant from CA and CB.

As this ⊙ must also touch BE, its centre lies on EO, the bisector of the $\angle CEB$.

Hence, the centre is at O, the intersection of CF and EO, and the radius is OK, the $\perp$ from O upon CE, SE, SD, or CD.

Construction. Draw the altitudes CF, BE, AD.

Bisect the $\angle CEB$ by EO, intersecting CF in O.

From O draw the $\perp$ OK to CA.

Then with O as centre and OK as radius describe a ⊙.

This ⊙ is inscribed in $ECDS$ and is one of the required Ⓢ.

The same construction may be repeated for each of the other quadrilaterals $EAFS$ and $FBDS$.

Proof. O is equidistant from CA and AB. § 162

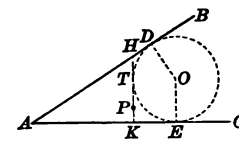
O is equidistant from EC and ES. § 162

Since $ECDS$ is symmetrical with respect to CS, § 212

O is equidistant from ES and DS. Q. E. F.

Ex. 601. *Given an angle and a point P between the sides of the angle. To draw through P a straight line which shall form with the sides of the angle a triangle with the perimeter equal to a given length a.*

Let BAC be the given $\angle$, and P the given point.



Construction. On AB take AD equal to $\frac{1}{2}a$, and on AC take AE equal to $\frac{1}{2}a$.

At D and E erect DO and $EO \perp$ to AB and AC, respectively.

With O as centre and OD as radius describe a $\odot$.

Through P draw HK tangent to the $\odot$ and terminated by AB and AC.

Then $\triangle AHK$ is the $\triangle$ required.

Proof. AB and AC are tangent to the $\odot$ at D and E, respectively. § 253

Let the tangent HK touch the $\odot$ at T.

Then $HD = HT$, and $KE = KT$. § 261

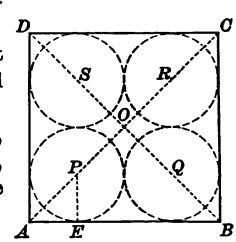
$\therefore AD + AE = AH + HT + AK + KT = AH + AK + HK.$

But $AD + AE = a.$ Const.

$\therefore AH + AK + HK = a.$ Q. E. F.

Ex. 602. *In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.*

Analysis. Suppose the problem solved, and let $ABCD$ be the given square and P, Q, R, S the centres of the required Ⓢ.



From the equality of the Ⓢ it is clear that their centres must be symmetrically arranged about O, the centre of the square.

Hence, we see that the $\measuredangle$ formed at O by OP, OQ, OR, and OS are each 90°, and, therefore, that P, Q, R, S lie in the diagonals of the square.

Construction. Draw the diagonals AC and BD, intersecting at O.

Bisect OA, OB, OC, OD at P, Q, R, S, respectively.

With P as centre and radius equal to the distance from P to the sides of the square describe a $\odot$. Construct similar Ⓢ with Q, R, S as centres.

These Ⓢ are the Ⓢ required.

Proof. The rt. $\triangle AEP$ and ABC are similar. § 356

$\therefore PE : BC = AP : AC = 1 : 4.$ § 351

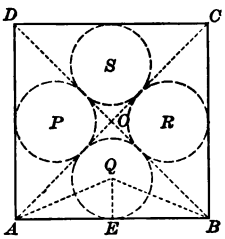
Therefore, each $\odot$ touches two others and two sides of the square.

Q. E. F.

Ex. 603. *In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to one side of the square.*

Let $ABCD$ be the given square.

To inscribe four equal Ⓢ in $ABCD$ so that each ⊙ shall be tangent to two of the others and also tangent to one side of the square.



Construction. Draw the diagonals AC and BD, intersecting at O.

The ◬ AOB, BOC, COD, DOA are equal isosceles ◬. § 143

For $OA = OB = OC = OD$, §§ 432, 217

and $\angle AOB = \angle BOC = \angle COD = \angle DOA$. § 136

Inscribe a ⊙ in each of these ◬.

These Ⓢ are the Ⓢ required.

Proof. From Q, the centre of one of these Ⓢ, draw QA and QB.

The $\triangle AQB$ is isosceles. § 147

For $\angle QAB = \angle QBA$, being halves of the equal $\measuredangle OAB$ and OBA. § 315

$\therefore$ the radius QE bisects AB. § 149

Since ◬ AOB and BOC are equal, and since only one ⊙ can be inscribed in a given $\triangle$, if the $\triangle BOC$ is folded over on OB as an axis, the $\triangle BOC$ will coincide with $\triangle BOA$ and R will coincide with Q.

$\therefore$ R and Q are symmetrical with respect to OB as an axis.

$\therefore$ RQ is $\perp$ to OB.

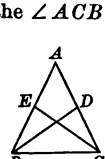
$\therefore$ the Ⓢ Q and R are tangent to each other.

In like manner, each ⊙ is tangent to two others. Q. E. F.

Proof for the converse theorem of Ex. 54 which was only stated in Ex. 56, page 16:

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

In the $\triangle ABC$ let BD bisect the $\angle ABC$ and CE bisect the $\angle ACB$; and let BD equal CE.

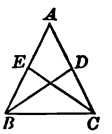


To prove that $AB = AC$.

Proof. The $\angle ABC$ is greater than, less than, or equal to the $\angle ACB$.

1. If the $\angle ABC$ is greater than the $\angle ACB$, the $\triangle ABC$ is not isosceles, and BD is not equal to CE. Ex. 55

2. If the $\angle ABC$ is less than the $\angle ACB$, the $\triangle ABC$ is not isosceles, and BD is not equal to CE. Ex. 55



But both these conclusions are contrary to the hypothesis that BD is equal to CE.

Therefore, since the $\angle ABC$ is not greater than or less than the $\angle ACB$,

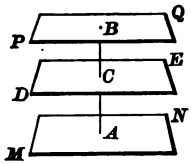
$$\angle ABC = \angle ACB.$$

$$\therefore AB = AC.$$ § 147

Q. E. D.

Ex. 604. *Find the locus of points in space equidistant from two given parallel planes.*

Let MN and PQ be any two $\parallel$ planes.



To find the locus of points equidistant from these planes.

Construction. At any point A in MN erect a $\perp$, and produce it to meet PQ in some point, as B.

Bisect AB, and through its middle point C pass a plane $DE \perp$ to AB.

The plane DE is the locus required.

Proof. The plane DE is $\parallel$ to MN. § 527

Hence, the planes MN, DE, and PQ are all parallel.

Any point in DE is at the distance AC from the plane MN.

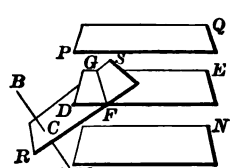
For like reason, any point in DE is at the distance BC from the plane PQ.

But, by construction, $AC = BC$; hence, any point in the plane DE is equally distant from MN and PQ.

Any point without the plane DE is evidently nearer one plane than the other; hence, the plane DE is the required locus. Q. E. F.

Ex. 605. *Find the locus of points in space equidistant from two given points, and also equidistant from two given parallel planes.*

Let A and B be any two given points in space, and MN and PQ any two given $\parallel$ planes.



To find the locus of points equidistant from A and B, and also equidistant from the planes MN and PQ.

Construction. Draw AB, bisect it; through C, its middle point, pass a plane $RS \perp$ to AB.

Construct the plane $DE \parallel$ to MN and PQ, and midway between them.

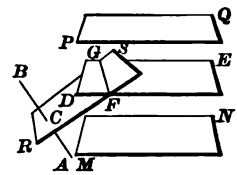
FG, the intersection of the planes RS and DE, is the locus required.

Proof. Every point in the plane RS is equidistant from A and B. § 517

Every point in the plane DE is equidistant from the planes MN and PQ. Ex. 604

Hence, every point in FG, the intersection of the planes RS and DE, is equidistant from the points A and B, and also from the planes MN and PQ. Therefore, FG is the required locus.

Q. E. F.

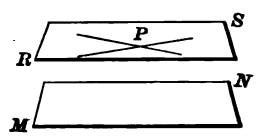


Discussion. If the line AB is $\perp$ to the plane MN, there will be no locus unless the middle point of AB is midway between the planes MN and PQ; in which case, the locus is the plane DE. In all other cases the locus is a straight line.

Ex. 606. *Find the locus of all lines drawn through a given point, parallel to a given plane.*

Let MN be the given plane and P the given point.

To find the locus of all lines drawn through $P \parallel$ to the plane MN.



Construction. Through P draw any two lines $\parallel$ to the plane MN.

Through these lines pass the plane RS.

The plane RS is the locus required.

Proof. The plane RS is $\parallel$ to the plane MN. § 533

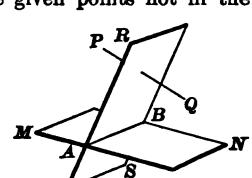
The plane RS is the only plane that can be passed through the point $P \parallel$ to the plane MN. § 532

Hence, the required locus is the plane through the given point parallel to the given plane. Q. E. F.

Ex. 607. *Find the locus of points in a given plane which are equidistant from two given points not in the plane.*

Let MN be the given plane and P, Q the given points not in the plane MN.

To find the locus of points in the plane MN equidistant from P and Q.



Construction. Draw PQ, and through the middle point of PQ pass the plane $RS \perp$ to PQ, intersecting the plane MN in the line AB.

The line AB is the locus required.

Proof. The plane RS is the locus of points in space equidistant from the points P and Q. § 517

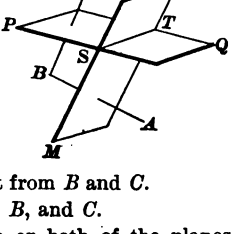
Therefore, AB, the intersection of the planes RS and MN, is the locus of points in the plane MN equidistant from P and Q. Q. E. F.

Discussion. If PQ is $\perp$ to MN, there is no locus. §§ 527, 504

Ex. 608. *Find the locus of a point in space equidistant from three given points not in a straight line.*

Let A, B, and C be any three points in space.

To find the locus of a point in space equidistant from A, B, and C.



Construction. Draw AB and BC. At the middle point of AB draw the plane $MN \perp$ to AB, and at the middle point of BC draw the plane $PQ \perp$ to BC. Let the line ST be the intersection of the planes MN and PQ.

Then ST is the locus required.

Proof. Since ST is in the plane MN, any point in ST is equidistant from A and B. § 517

For like reason, any point in ST is equidistant from B and C.

Hence, any point in ST is equidistant from A, B, and C.

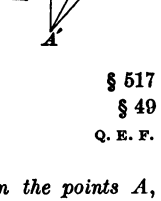
Any point without this line is without one or both of the planes MN and PQ, and is therefore not equidistant from the points A, B, and C.

Hence, ST is the locus required. Q. E. F.

Ex. 609. *Find a point in a plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.*

Let MN be the given plane, and A and B the given points on the same side of MN.

To find a point in MN such that the sum of its distances from A and B shall be a minimum.



Construction. From A draw $AD \perp$ to the plane MN, and produce it to A', making DA' equal to AD. Draw $A'B$, cutting MN in C, and draw AC.

Then $AC + BC$ is less than the sum of any other two lines drawn from A and B to any point in MN.

Proof. Let F be any other point than C of the plane MN. Draw FB, FA, and FA'.

Now	$FA = FA'$, and $CA = CA'$.	§ 517
But	$BF + FA' > BC + CA'$.	§ 49
	$\therefore BF + FA > BC + CA$.	Q. E. F.

Ex. 610. *The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE = 6$ inches, $BE = 8$ inches, $CD = 12$ inches, compute CF and FD.*

$AE : EB = CF : FD.$ § 535

$\therefore AE + EB : EB = CF + FD : FD.$ § 332

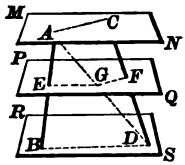
That is, $AB : EB = CD : FD.$

$14 : 8 = 12 : FD.$

$$FD = \frac{8 \times 12}{14} = \frac{48}{7} = 6\tfrac{6}{7}.$$

$$CF = CD - FD = 12 - 6\tfrac{6}{7} = 5\tfrac{1}{7}.$$

$\therefore CF = 5\frac{1}{7}$ inches; $FD = 6\frac{6}{7}$ inches. *Ans.*



Ex. 611. *To draw a perpendicular to a given plane from a given point without the plane.*

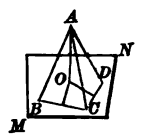
Let MN be the given plane and A the given point without the plane.

To draw a ⊥ from A to the plane MN.

Construction. From A draw any three equal lines to the plane MN, as AB, AC, and AD.

In the plane MN find the point O equidistant from the points B, C, D. Draw AO.

Then AO is the ⊥ required.



Proof. The foot of the ⊥ from A to the plane MN is equidistant from the feet of the equal oblique lines AB, AC, and AD. § 515

But O is the only point in MN equidistant from B, C, and D. § 314

Hence, AO is the ⊥ from A to MN. § 511

Q. E. F.

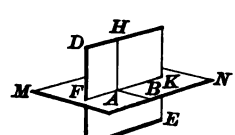
Ex. 612. *To erect a perpendicular to a given plane at a given point in the plane.*

Let A be any point in the given plane MN.

To erect a ⊥ to the plane MN at the point A.

Construction. Through A draw any line BA in the plane MN.

Through A pass the plane ED ⊥ to BA, cutting MN in FAK.



In the plane ED draw AH ⊥ to FK.

Then AH is the ⊥ required.

Proof. Since BA is ⊥ to the plane ED, it is ⊥ to AH. § 501

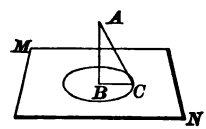
But AH is ⊥ to FK. Const.

$\therefore AH$ is ⊥ to BA and FK of the plane MN at their intersection A.

$\therefore AH$ is ⊥ to the plane MN at A. § 507

Q. E. F.

Ex. 613. *From a point A, 4 inches from a plane MN, an oblique line AC, 5 inches long, is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.*



AB is $\perp$ to MN and, therefore, $\perp$ to BC. § 501

$\therefore \overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$. § 372

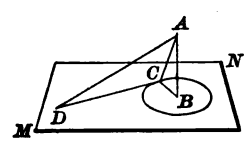
$\therefore BC = \sqrt{\overline{AC}^2 - \overline{AB}^2} = \sqrt{5^2 - 4^2} = 3.$

Area $\odot = \pi R^2 = (3.1416 \times 3^2)$ sq. in. § 463

$= 28.2744$ sq. in.

Ex. 614. *From a point A, 8 inches from a plane MN, a perpendicular AB is drawn to the plane; with B as centre, and a radius equal to 6 inches, a circle is described in the plane; at any point C of this circumference a tangent CD, 24 inches long, is drawn. Find the distance from A to D.*

Let AB, 8 inches long, be $\perp$ to the plane MN. Let a circumference be described with B as centre and radius BC equal to 6 inches, and let DC 24 inches long be tangent to this circumference at C.



To find the length AD.

Now AB is $\perp$ to BC. § 501

$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ § 371

$= 8^2 + 6^2 = 100.$

Now AB is $\perp$ to BC, § 501

and BC is $\perp$ to DC. § 254

$\therefore AC$ is $\perp$ to CD. § 518

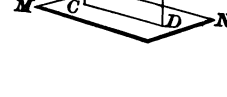
$\therefore \overline{AD}^2 = \overline{AC}^2 + \overline{CD}^2$ § 371

$= 100 + 24^2 = 100 + 576 = 676.$

$\therefore AD = \sqrt{676}$ in. $= 26$ in.

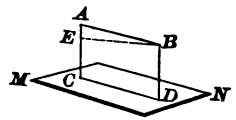
Ex. 615. *Describe the relative position to a given plane of a line if its projection on the plane is equal to its own length.*

Let CD be the projection of the line AB on the plane MN, and let CD be equal to AB. AB is $\parallel$ to CD and, therefore, $\parallel$ to the plane MN. Otherwise, either A or B would be nearer the plane than the other. Suppose $BD < AC$.



Draw $BE \perp$ to AC. Then BE and DC, being both $\perp$ to AC, would be $\parallel$. § 104

$\therefore BE$ and DC would be equal. § 180
But $BE < BA$. § 97
$\therefore DC < BA$.



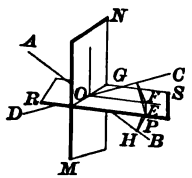
But this is contrary to the hypothesis that BA and DC are equal.

$\therefore AB$ is $\parallel$ to CD, and, therefore, to the plane MN.

Ex. 616. *Find the locus of points in space equidistant from two given intersecting lines.*

Let AB and CD be two lines in the plane of the paper intersecting at O.

To find the locus of points in space equidistant from AB and CD.



Construction. Draw OE in the plane of AB and CD bisecting the $\angle COB$. Through OE pass the plane $RS \perp$ to the plane of AB and CD.

The plane RS is one locus required.

Proof. From P, any point in the plane RS, draw $PF \perp$ to OE.

PF is also $\perp$ to the plane of AB and CD. § 551

Draw FG and $FH \perp$ to CD and AB, respectively. Draw PG and PH.

PG and PH are $\perp$ to CD and AB, respectively. § 518

Now the rt. $\triangle PFG$ and PFH are equal. § 144

For $PF = PF$, Iden.

and $FG = FH$. § 162

$\therefore PG = PH$. § 128

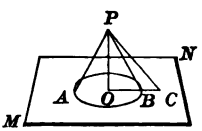
That is, P is equidistant from CD and AB.

In like manner, another locus is the plane MN passed through the bisector of the $\angle AOC \perp$ to the plane of the lines AB and CD. Q. E. F.

Ex. 617. *Find the locus of points in space equidistant from all points in the circumference of a circle.*

Let AB be the circumference of a $\odot$ with centre O lying in the plane MN.

To find the locus of points in space equidistant from all points in the circumference of the circle.



Construction. At O erect the $\perp OP$ to the plane MN.

Then OP is the locus required.

Proof. From any point P of this $\perp$ draw PA and PB to any points of the circumference, and PC to any point of the plane MN not in the circumference.

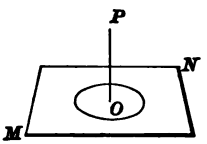
Now PA and PB are equal, § 514

and PA and PC are unequal. § 514

Therefore PO is the required locus. Q. E. F.

Ex. 618. *Find the locus of points in a plane equidistant from a given point without the plane.*

Let P be the given point and MN the given plane.



To find the locus of points in the plane MN equidistant from the point P.

Construction. Draw $PO \perp$ to the plane MN.

With O as centre describe any circumference.

This circumference is the locus required.

Proof. All lines drawn from P to this circumference are equal. § 514

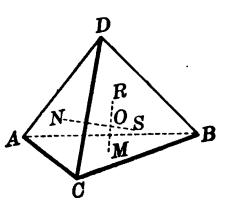
Q. E. F.

Discussion. Since with O for centre any number of circumferences may be described, the locus consists of an indefinite number of circumferences.

Ex. 619. *Find a point at equal distances from four points not all in the same plane.*

Let A, B, C, D be the four given points not all in the same plane.

To find a point equidistant from A, B, C, D.



Construction. Draw AB, AC, BC, AD, CD, BD.

Find M, N, the centres, respectively, of the ⊙ circumscribed about the △ ABC, ACD.

Draw $MR \perp$ to the plane ABC and $NS \perp$ to the plane ACD.

MR is the locus of points equidistant from A, B, and C; and NS is the locus of points equidistant from A, C, and D. § 516

Also MR and NS lie in the same plane.

For, if a plane $\perp$ to AC is passed through its middle point, this plane will contain all points equidistant from A and C. § 517

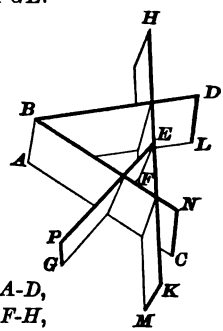
$\therefore$ MR and NS must lie in this plane.

Again, MR and NS, being $\perp$ to planes which are not ∥, cannot be ∥, and must, therefore, meet at some point O.

$\therefore$ O is equidistant from A, B, C, D. Q. E. F.

Ex. 620. *Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.*

Let the plane HM be $\perp$ to the plane AD, and let the plane GE be $\perp$ to the plane AN. Let AB, the intersection of the planes AN and AD, be $\parallel$ to FE, the intersection of the planes HM and GE.



To prove that the dihedral $\angle C\text{-}BA\text{-}D$ is equal to the dihedral $\angle G\text{-}EF\text{-}K$ and is supplementary to the dihedral $\angle G\text{-}EF\text{-}H$.

Proof. Pass the plane $BHK \perp$ to the line AB, cutting the planes CB, AD, HM, and GE in the lines BN, BD, HK, EP.

The plane BHK is $\perp$ to the line FE. § 520

NB and DB are $\perp$ to AB at the same point B, and HK and PE are $\perp$ to FE at the same point E.

§ 501

$\therefore \angle NBD$ is the measure of the dihedral $\angle C\text{-}BA\text{-}D$,

$\angle HEP$ is the measure of the dihedral $\angle G\text{-}EF\text{-}H$,

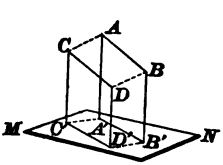
and $\angle PEK$ is the measure of the dihedral $\angle G\text{-}EF\text{-}K$. § 550

But $\angle NBD$ is equal to $\angle PEK$ and supplementary to $\angle HEP$. Ex. 90

$\therefore$ dihedral $\angle C\text{-}BA\text{-}D$ is equal to dihedral $\angle G\text{-}EF\text{-}K$ and is supplementary to dihedral $\angle G\text{-}EF\text{-}H$. Q. E. D.

Ex. 621. *The projections on a plane of equal and parallel lines are equal and parallel.*

Let AB and CD be equal and parallel lines, and let $A'B'$ and $C'D'$ be the projections of AB and CD on any plane MN.



To prove that $A'B'$ and $C'D'$ are equal and parallel.

Proof. Draw AC and BD.

AB and CD lie in one plane. § 498

Now AB and CD are equal and parallel. Hyp.

$\therefore ABDC$ is a $\square$ and AC and BD are equal and parallel. § 183

The planes AB', CD', AC', and BD' are $\perp$ to the plane MN. § 554

The planes AB' and CD' are $\parallel$, otherwise AB and CD, the intersections of these planes with the plane $ABCD$, could be made to meet, which is contrary to the hypothesis that AB and CD are $\parallel$.

In like manner, the planes AC' and BD' are parallel.

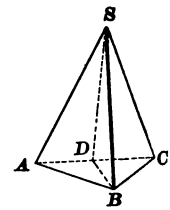
$\therefore A'B'$ and $C'D'$ are $\parallel$. § 528

$\therefore A'B'$ and $C'D'$ are equal. § 529

Q. E. D.

Ex. 622. *If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.*

In the trihedral $\angle S\text{-}ABC$, let the face $\measuredangle ASB$ and BSC be equal.



To prove that the dihedral $\measuredangle SA$ and SC are equal.

Proof. Draw SD bisecting the face $\angle ASC$.

Pass the plane BSD through SB and SD.

Then the trihedral $\measuredangle S\text{-}ABD$ and $S\text{-}CBD$ are symmetrical. § 582

For $\angle ASB = \angle CSB$, Hyp.

$\angle BSD = \angle BSD$, Iden.

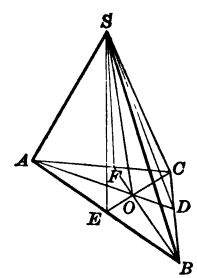
and $\angle DSA = \angle DSC$. Const.

$\therefore$ dihedral $\angle SA$ = dihedral $\angle SC$. § 579

Q. E. D.

Ex. 623. *The planes that bisect the dihedral angles of a trihedral angle intersect in the same straight line.*

Let $S\text{-}ABC$ be a trihedral $\angle$, and let the planes SAD, SBF, and SCE bisect the dihedrals SA, SB, and SC, respectively.



To prove that the planes SAD, SBF, SCE intersect in the same straight line.

Proof. Let SAD and SBF intersect in the straight line SO.

Then every point in SO is equally distant from the planes SAC and SAB. § 559

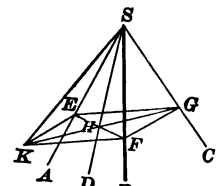
For like reason, every point in SO is equally distant from the planes SBA and SBC.

Hence, every point in SO is equally distant from the planes SCB and SCA.

Therefore, SO is in the bisector of the dihedral SC.

Hence, the planes SAD, SBF, SCE intersect in the same straight line SO. Q. E. D.

Ex. 624. *If the face angle ASB of the trihedral angle $S\text{-}ABC$ is bisected by the line SD, the angle CSD is less than half the sum of the angles ASC and BSC.*



Let SD bisect the face $\angle ASB$ of the trihedral $\angle S\text{-}ABC$.

To prove that $\angle CSD < \frac{1}{2}(\angle ASC + \angle BSC)$.

Proof. Through H, any point in SD, pass a plane $\perp$ to SD, cutting the edges SA, SB, SC in the points E, F, G, respectively.

Since SH is $\perp$ to the plane EFG, it is $\perp$ to EF. § 501

The rt. $\triangle$ SHE and SHF are equal. § 142

For SH is common,

and $\angle ESH = \angle FSH$. Hyp.

$\therefore SE = SF$ and $HE = HF$. § 128

Produce GH to K, making HK equal to GH, and draw SK, EK, FK.

Then $FGEK$ is a $\square$. Ex. 68

$\therefore KE = FG$. § 178

Then $\triangle SEK$ and SFG are equal. § 150

For $KE = FG$, $SE = SF$,

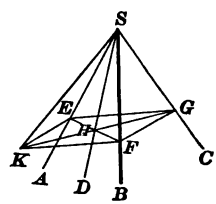
and $SK = SG$. § 160

$\therefore \angle ESK = \angle FSG$. § 128

Now $\angle KSH = \angle GSH$. § 95

$\therefore \angle GSK = 2\angle GSH$

$= 2\angle CSD$.



In the trihedral angle S-EKG,

$\angle GSK$ is less than $\angle ESG + \angle ESK$. § 580

That is, $2\angle CSD$ is less than $\angle ASC + \angle BSC$.

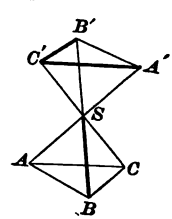
$\therefore \angle CSD$ is less than $\frac{1}{2}(\angle ASC + \angle BSC)$. Ax. 7

Q. E. D.

Ex. 625. *An isosceles trihedral angle and its symmetrical trihedral angle are superposable.*

Let S-ABC be an isosceles trihedral $\angle$, having $\angle ASB$ equal to $\angle BSC$, and let S-$A'B'C'$ the symmetrical trihedral $\angle$.

To prove that the trihedral $\measuredangle$ are superposable.



Proof. $\angle ASB = \angle BSC$. Hyp.

$\angle C'SB' = \angle BSC$. § 93

$\therefore \angle ASB = \angle C'SB'$. Ax. 1

For like reason, $\angle BSC = \angle B'SA'$.

Dihedral A-BS-C = dihedral C'-$B'S$-A'. § 547

Revolve S-$A'B'C'$ about S until SB' falls on SB, and the equal dihedrals SB' and SB coincide.

Then, since $\angle B'SA' = \angle BSC$, SA' will coincide with SC.

For like reason, SC' will fall on SA.

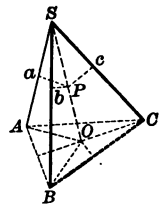
Hence, the edges of the trihedral $\measuredangle$ coincide.

Therefore, the trihedral $\measuredangle$ coincide. Q. E. D.

Ex. 626. *Find the locus of points equidistant from the three edges of a trihedral angle.*

Let S-ABC be any trihedral $\angle$.

To find the locus of points equidistant from the three edges SA, SB, and SC.



Construction. Lay off SA equal to SB equal to SC, and draw the lines AB, BC, and AC.

In the plane ABC find the point O equally distant from A, B, and C. § 314

Draw SO.

Then SO is the locus required.

Proof. Since O is equally distant from A, B, and C, and $SA = SB = SC$, and OS is common, the $\triangle$ OSA, OSB, and OSC are mutually equilateral and equal. § 150

Hence, the $\measuredangle$ OSA, OSB, and OSC are equal. § 128

From P, any point in SO, draw the $\perp$s Pa, Pb, and Pc to the edges SA, SB, and SC, respectively.

The rt. $\triangle$ SaP, SbP, and ScP are equal, since the side SP is common, and the $\measuredangle$ at S are equal. § 141

$\therefore Pa = Pb = Pc.$ § 128

$\therefore P$ is equally distant from the three edges SA, SB, and SC.

The line drawn from S to any point without SO would not pierce the plane ABC at a point equidistant from A, B, and C, and, therefore, would not make equal $\measuredangle$ with the edges. § 155

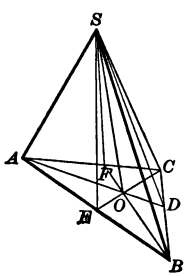
$\therefore$ the $\perp$s from any point of this line to the edges would not be equal. § 154

Therefore, SO is the locus of all points equidistant from the edges SA, SB, and SC. Q. E. F.

Ex. 627. *Find the locus of points equidistant from the three faces of a trihedral angle.*

Let S-ABC be any trihedral $\angle$.

To find the locus of points equidistant from the three faces SAB, SBC, and SAC.



Construction. Draw the plane SBF, bisecting the dihedral $\angle SB$; also draw the plane SAD, bisecting the dihedral $\angle SA$. Let SO be the intersection of these bisecting planes.

Then SO is the locus required.

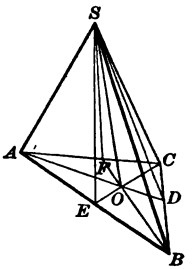
Proof. Since any point in SO is in the plane SBF, it is equally distant from the faces SBA and SBC. § 559

For like reason, any point in SO is equally distant from the faces SAB and SAC.

Hence, any point in SO is equally distant from the faces SAB, SBC, and SAC.

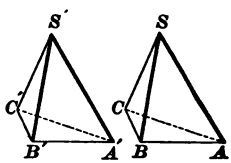
Any point without SO is without one or both of the planes SBF and SAD, and is, therefore, not equally distant from the faces SAB, SBC, and SAC.

Hence, SO is the locus required. Q. E. F.



Ex. 628. *Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal, respectively, to two dihedral angles and the included face angle of the other, and similarly placed.*

In the trihedral $\measuredangle$ S-ABC and S'-$A'B'C'$, let the dihedral $\measuredangle$ SA, SB, and the face $\angle ASB$ be equal, respectively, to the dihedral $\measuredangle$ $S'A'$, $S'B'$, and the face $\angle A'S'B'$, and similarly placed.



To prove that the trihedral $\measuredangle$ S-ABC and S'-$A'B'C'$ are equal.

Proof. Place the trihedral $\angle$ S-ABC on the trihedral $\angle$ S'-$A'B'C'$ so that the face $\angle ASB$ shall coincide with its equal, the face $\angle A'S'B'$.

Then the plane SBC will fall on the plane $S'B'C'$, since the dihedral $\angle SB$ = the dihedral $\angle S'B'$. Hyp.

Also, the plane ASC will fall on the plane $A'S'C'$, since the dihedral $\angle SA$ = the dihedral $\angle S'A'$. Hyp.

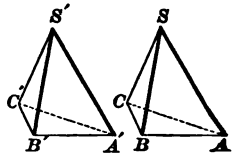
Then SC, the intersection of the planes SBC and ASC, must lie in both the planes $S'B'C'$ and $A'S'C'$ and, therefore, must coincide with $S'C'$, the intersection of the planes $S'B'C'$ and $A'S'C'$.

Therefore, the trihedral $\measuredangle$ coincide and are equal. § 577

Q. E. D.

Ex. 629. *Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal, respectively, to two face angles and the included dihedral angle of the other, and similarly placed.*

In the trihedral $\measuredangle$ S-ABC and S'-$A'B'C'$, let the face $\measuredangle$ ASB, BSC, and the dihedral $\angle SB$ be equal, respectively, to the face $\measuredangle$ $A'S'B'$, $B'S'C'$, and the dihedral $\angle S'B'$, and similarly placed.



To prove that the trihedral $\measuredangle$ S-ABC and S'-$A'B'C'$ are equal.

Proof. Place the trihedral $\angle S\text{-}ABC$ on the trihedral $\angle S'\text{-}A'B'C'$ so that the vertex S shall fall on the vertex S', and the dihedral $\angle SB$ shall fall on the dihedral $\angle S'B'$.

The face $\measuredangle ASB$ and $A'S'B'$ lie in the same plane $A'S'B'$.

Then SA falls on $S'A'$, since the $\measuredangle ASB$ and $A'S'B'$ are equal. Hyp.

The face $\measuredangle BSC$ and $B'S'C'$ lie in the same plane $B'S'C'$.

Then SC falls on $S'C'$, since the $\measuredangle BSC$ and $B'S'C'$ are equal. Hyp.

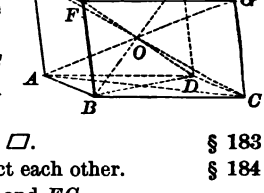
Hence, the face $\measuredangle ASC$ and $A'S'C'$ coincide. § 497

Therefore, the trihedral $\measuredangle$ coincide and are equal. § 577

Q. E. D.

Ex. 630. *The diagonals of a parallelopiped bisect one another.*

Let AG, EC, BH, and FD be the four diagonals of the parallelopiped AG.



To prove that the four diagonals bisect one another.

Proof. Through the opposite and ‖ edges AE and CG pass a plane intersecting the ‖ bases in the ‖ lines AC and EG.

The section $ACGE$ is a $\square$. § 183

$\therefore$ its diagonals AG and EC bisect each other. § 184

That is, O is the middle point of AG and EC.

In like manner, a plane passed through the opposite and ‖ edges FG and AD will form a $\square$ $AFGD$.

Hence, O, the middle point of the diagonal AG, is also the middle point of FD. § 184

Also, a plane passed through the opposite and ‖ edges EH and BC will form a $\square$ $EBCH$.

Hence, O, the middle point of the diagonal EC, is also the middle point of BH. § 184

Hence, the four diagonals bisect one another at the point O. Q. E. D.

Ex. 631. *The lateral faces of a right prism are rectangles.*

Proof. The lateral edges of any right prism are $\perp$ to the planes of its bases. § 591

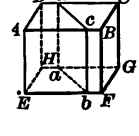
Hence, these lateral edges are $\perp$ to the sides of the bases of the prism, or the $\measuredangle$ of the lateral faces are rt. $\measuredangle$. § 501

That is, these faces are rectangles. § 167

Q. E. D.

Ex. 632. *Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.*

Let $abcd$ be a section of the prism AG, made by the plane abc ‖ to the lateral edges AE, BF, CG, and DH.



To prove that $abcd$ is a $\square$.

Proof. The lines ab and dc are parallel. § 528

Since the plane *abc* and the edge *BF* are ||, the lines *bc* and *FB* cannot meet, however far they are produced; hence, as they lie in the same plane *ABF*, these lines are parallel.

For like reason, *ad* and *HD* are parallel.

But *DH* and *BF* are parallel.

$\therefore$ *bc* and *ad* are parallel. § 521

Therefore, *abcd* is a $\square$. § 166

Q. E. D.

Ex. 633. *Find the lateral area of a right prism, if its altitude is* 18 *inches and the perimeter of its base* 29 *inches.*

$S = E \times P = (18 \times 29)$ sq. in. $= 522$ sq. in. § 608

Ex. 634. *Find the lateral area of a right prism, if its altitude is* 20 *inches and its base is a triangle whose sides are* 7 *inches,* 8 *inches, and* 9 *inches, respectively.*

$S = E \times P = 20 \times (7 + 8 + 9) = 20 \times 24 = 480.$ § 608

480 sq. in. *Ans.*

Ex. 635. *Find the lateral area of a triangular prism, if its lateral edge is* 20 *inches, and its right section is a triangle whose sides are* 9 *inches,* 10 *inches, and* 12 *inches, respectively.*

$S = E \times P = 20 \times (9 + 10 + 12) = 20 \times 31 = 620.$ § 607

620 sq. in. *Ans.*

Ex. 636. *Find the total area of a right prism, if its altitude is* 32 *inches, and its base is a triangle whose sides are* 12 *inches,* 14 *inches, and* 16 *inches, respectively.*

$S = E \times P = 32 \times (12 + 14 + 16) = 32 \times 42 = 1344.$ § 608

Area of each base $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ Ex. 405

$= \sqrt{21 \times 9 \times 7 \times 5}$

$= 81.33.$

Total area $= 1344$ sq. in. $+ 2 \times 81.33$ sq. in. $= 1506.66$ sq. in.

$= 10$ sq. ft. 66.66 sq. in.

Ex. 637. *Find the ratio of two rectangular parallelopipeds, if their altitudes are each* 6 *inches, and their bases* 5 *inches by* 4 *inches, and* 10 *inches by* 8 *inches, respectively.*

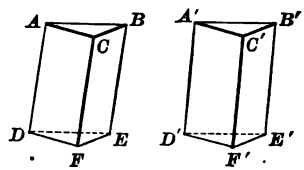
$$\frac{P}{P'} = \frac{5 \times 4}{10 \times 8} = \frac{1}{4}.$$ § 619

Ex. 638. *Find the ratio of two rectangular parallelopipeds, if their dimensions are* 3, 4, 5, *and* 9, 8, 10, *respectively.*

$$\frac{P}{P'} = \frac{3 \times 4 \times 5}{9 \times 8 \times 10} = \frac{1}{12}. \qquad \text{§ 621}$$

Ex. 639. *Two triangular prisms are equal if their lateral faces are equal, each to each, and similarly placed.*

Let $ABC\text{-}DEF$ and $A'B'C'\text{-}D'E'F'$ be triangular prisms having their lateral faces AF and $A'F'$, CE and $C'E'$, AE and $A'E'$ equal, respectively, and similarly placed.



To prove that

$$ABC\text{-}DEF = A'B'C'\text{-}D'E'F'.$$

Proof. The $\triangle ABC$ and $A'B'C'$ are equal. § 150

For $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$, being homologous sides of equal polygons.

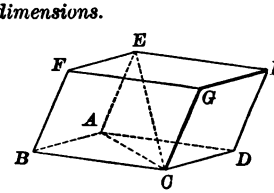
$\therefore ABC\text{-}DEF = A'B'C'\text{-}D'E'F'$. § 609

For face AE = face $A'E'$ and face AF = face $A'F'$, Hyp.

and base ABC = base $A'B'C'$. Q. E. D.

Ex. 640. *The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three dimensions.*

Let CE be a diagonal, and CD, CB, and CG the dimensions of the rectangular parallelopiped $E\text{-}ABCD$.



To prove that

$$\overline{CE}^2 = \overline{CD}^2 + \overline{CB}^2 + \overline{CG}^2.$$

Proof. Draw CA.

Then the $\triangle ACD$ and ACE are rt. $\triangle$.

Hence, $\overline{CA}^2 = \overline{CD}^2 + \overline{AD}^2 = \overline{CD}^2 + \overline{CB}^2$, § 371

and $\overline{CE}^2 = \overline{CA}^2 + \overline{AE}^2 = \overline{CA}^2 + \overline{CG}^2$. § 371

$\therefore \overline{CE}^2 = \overline{CD}^2 + \overline{CB}^2 + \overline{CG}^2$. Q. E. D.

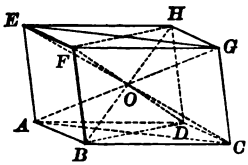
Ex. 641. *The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.*

Let AG, CE, BH, DF be the four diagonals of the parallelopiped AG.

To prove that $\overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 + \overline{BF}^2 + \overline{CG}^2 + \overline{DH}^2 + \overline{AE}^2 + \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$.

Proof. Draw AC, EG, FH, and BD.

Since the sections AG and BH are $\mathcal{S}$, Ex. 630
$\overline{AG}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{CG}^2 + \overline{EG}^2 + \overline{AC}^2$,
Ex. 280
and $\overline{BH}^2 + \overline{DF}^2 = \overline{BF}^2 + \overline{DH}^2 + \overline{FH}^2 + \overline{BD}^2$.
Ex. 280



$\therefore \overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2$
$= \overline{AE}^2 + \overline{CG}^2 + \overline{BF}^2 + \overline{DH}^2 + \overline{EG}^2 + \overline{FH}^2 + \overline{AC}^2 + \overline{BD}^2$. Ax. 2.

Now, since the bases EG and AC are $\mathcal{S}$, Hyp.

$\overline{EG}^2 + \overline{FH}^2 = \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$, Ex. 280

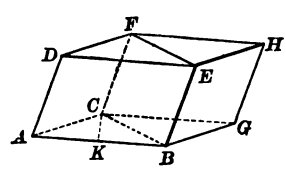
and $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2$. Ex. 280

Substitute these values for $\overline{EG}^2 + \overline{FH}^2$ and $\overline{AC}^2 + \overline{BD}^2$.

Then $\overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 + \overline{BF}^2 + \overline{CG}^2 + \overline{DH}^2 + \overline{AE}^2 + \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$. Q. E. D.

Ex. 642. *The volume of a triangular prism is equal to half the product of any lateral face by the perpendicular dropped from the opposite edge on that face.*

Let $ABC\text{-}DEF$ be a triangular prism, and let CK be a $\perp$ dropped from any point C of the edge CF to the plane of the face AE.



To prove that volume
$ABC\text{-}DEF = \frac{1}{2}\, AE \times CK$.

Proof. Upon the edges AB, AC, AD construct the parallelopiped AH.

Then prism $ABC\text{-}DEF = \frac{1}{2}$ parallelopiped AH. § 615

But volume parallelopiped $AH = AE \times CK$. § 626

$\therefore$ volume prism $ABC\text{-}DEF = \frac{1}{2}\, AE \times CK$. Q. E. D.

Ex. 643. *If the edge of a cube is* 15 *inches, find the area of the total surface of the cube.*

Area of each face $= (15 \times 15)$ sq. in. § 398

Hence, total area $= 6 \times (15 \times 15)$ sq. in. $= 1350$ sq. in.

Ex. 644. *If the length of a rectangular parallelopiped is* 10 *inches, its width* 8 *inches, and its height* 6 *inches, find the area of its total surface.*

Perimeter of base $= 2 \times (10$ in. $+ 8$ in.$) = 36$ in.

Lateral area $= (6 \times 36)$ sq. in. $= 216$ sq. in. § 608

Area of bases $= 2 \times (10 \times 8)$ sq. in. $= 160$ sq. in. § 398

$\therefore$ total area $= 216$ sq. in. $+ 160$ sq. in. $= 376$ sq. in.

Ex. 645. *Find the volume of a right triangular prism, if its height is* 14 *inches, and the sides of the base are* 6, 5, *and* 5 *inches.*

Area of base $= \sqrt{8 \times 2 \times 3 \times 3}$ sq. in. $= 12$ sq. in. Ex. 405

$V = (14 \times 12)$ cu. in. $= 168$ cu. in. § 628

Ex. 646. *The base of a right prism is a rhombus, one side of which is* 10 *inches, and the shorter diagonal is* 12 *inches. The height of the prism is* 15 *inches. Find the entire surface and the volume.*

$\frac{1}{2}$ longer diagonal of base $= \sqrt{10^2 - 6^2} = 8.$ § 372

Base $= \frac{1}{2}(16 \times 12) = 96.$ Ex. 359

$S = 4 \times 10 \times 15 = 600.$ § 608

Bases $= 2 \times 96 = 192.$

$T = 600$ sq. in. $+ 192$ sq. in. $= 792$ sq. in.

$V = (15 \times 96)$ cu. in. $= 1440$ cu. in. § 626

Ex. 647. *Find the volume of a regular prism whose height is* 10 *feet, if each side of its triangular base is* 10 *inches.*

$$\text{Base} = \frac{10^2\sqrt{3}}{4} \text{ sq. in.} = 25\sqrt{3} \text{ sq. in.} = 43.3 \text{ sq. in.} = \frac{43.3}{144} \text{ sq. ft.} \quad \text{Ex. 404}$$

$$V = \left(10 \times \frac{43.3}{144}\right) \text{cu. ft.} = \frac{433}{144} \text{ cu. ft.} = 3 \text{ cu. ft.} \quad \text{§ 628}$$

Ex. 648. *How many square feet of lead will be required to line a cistern, open at the top, which is* 4 *feet* 6 *inches long,* 2 *feet* 8 *inches wide, and contains* 42 *cubic feet?*

4 ft. 6 in. $= 4\frac{1}{2}$ ft.; 2 ft. 8 in. $= 2\frac{2}{3}$ ft.

$$\text{Height} = \frac{42}{4\frac{1}{2} \times 2\frac{2}{3}} \text{ ft.} = 3\tfrac{1}{2} \text{ ft.}$$

$S = 3\frac{1}{2} \times 2 \times (4\frac{1}{2} + 2\frac{2}{3})$ sq. ft. $= 50\frac{1}{6}$ sq. ft. § 608

$B = (4\frac{1}{2} \times 2\frac{2}{3})$ sq. ft. $= 12$ sq. ft. § 398

$50\frac{1}{6}$ sq. ft. $+ 12$ sq. ft. $= 62\frac{1}{6}$ sq. ft.

Ex. 649. *An open cistern* 6 *feet long and* $4\frac{1}{2}$ *feet wide holds* 108 *cubic feet of water. How many square feet of lead will it take to line the sides and bottom?*

$B = (6 \times 4\frac{1}{2})$ sq. ft. $= 27$ sq. ft. § 398

Height $= \frac{108}{27}$ ft. $= 4$ ft.

$S = 4 \times 2 \times (6 + 4\frac{1}{2}) = 84$ sq. ft. § 608

27 sq. ft. $+ 84$ sq. ft. $= 111$ sq. ft.

Ex. 650. *An open cistern is made of iron 2 inches thick. The inner dimensions are: length, 4 feet 6 inches; breadth, 3 feet; depth, 2 feet 6 inches. What will the cistern weigh (i) when empty? (ii) when full of water?* (*The specific gravity of the iron is* 7.2.)

Outside length $= 4$ ft. 6 in. $+ 4$ in. $= 4$ ft. 10 in. $= 4\frac{5}{6}$ ft.
Outside width $= 3$ ft. $+ 4$ in. $= 3$ ft. 4 in. $= 3\frac{1}{3}$ ft.
Volume of bottom $= (4\frac{5}{6} \times 3\frac{1}{3} \times \frac{1}{6})$ cu. ft. $= 2\frac{37}{54}$ cu. ft.
Volume of sides $= 2 \times (4\frac{5}{6} \times 2\frac{1}{2} \times \frac{1}{6})$ cu. ft. $= 4\frac{1}{36}$ cu. ft.
Volume of ends $= 2 \times (3 \times 2\frac{1}{2} \times \frac{1}{6})$ cu. ft. $= 2\frac{1}{2}$ cu. ft.
$2\frac{37}{54}$ cu. ft. $+ 4\frac{1}{36}$ cu. ft. $+ 2\frac{1}{2}$ cu. ft. $= 9\frac{23}{108}$ cu. ft.
Weight when empty $= 9\frac{23}{108} \times 7\frac{1}{5} \times 62\frac{1}{2}$ lb. $= 4145\frac{5}{8}$ lb.
Weight of water $= 4\frac{1}{2} \times 3 \times 2\frac{1}{2} \times 62\frac{1}{2}$ lb. $= 2109\frac{3}{8}$ lb.
Weight when full $= 4145\frac{5}{8}$ lb. $+ 2109\frac{3}{8}$ lb. $= 6255\frac{5}{24}$ lb.

Ex. 651. *Find the volume of a right hexagonal prism, if its height is* 10 *feet, and each side of the hexagon is* 10 *inches.*

$$a = \sqrt{10^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75} = 8.66025.$$

Base $= 6 \times \frac{1}{2}(10 \times 8.66025)$ sq. in $= 259.8075$ sq. in. § 459
Altitude $= 10$ ft. $= 120$ in.
$V = (120 \times 259.8075)$ cu. in. $= 31176.9$ cu. in. § 628
$= 18$ cu. ft. 73 cu. in.

Ex. 652. *Find the length of an edge of a cubical vessel that will hold* 2 *tons of water.*

2 t. $= 2 \times 2000$ lb. $= 4000$ lb.

The volume of water required to weigh 4000 lb. is

$$\frac{4000}{62\frac{1}{2}} \text{ cu. ft.} = 64 \text{ cu. ft.}$$

$\therefore$ the edge of the cube $= \sqrt[3]{64}$ ft. $= 4$ ft.

Ex. 653. *One edge of a cube is* a. *Find the surface, the volume, and the length of a diagonal of the cube.*

$$T = 6 \times a \times a = 6a^2.$$
$$V = a \times a \times a = a^3.$$

If d denotes the diagonal of the cube,

$$4d^2 = 12a^2. \qquad \text{Ex. 641}$$
$$\therefore d^2 = 3a^2.$$
$$\therefore d = a\sqrt{3}.$$

Ex. 654. *A diagonal of one of the faces of a cube is a. Find the volume of the cube.*

Let x = the edge of the cube.

Then $a^2 = x^2 + x^2 = 2x^2$. § 372

$\therefore x^2 = \frac{a^2}{2}$.

$\therefore x = \frac{1}{2}a\sqrt{2}$.

$V = x^3 = (\frac{1}{2}a\sqrt{2})^3 = \frac{1}{4}a^3\sqrt{2}$. § 623

Ex. 655. *The three dimensions of a rectangular parallelopiped are a, b, c. Find the area of its surface, its volume, and the length of a diagonal.*

$T = 2ab + 2ac + 2bc = 2(ab + ac + bc)$.

$V = abc$. § 622

$\therefore$ diagonal $= \sqrt{a^2 + b^2 + c^2}$. Ex. 640

Ex. 656. *The volume of a parallelopiped is V, and the three dimensions are as $m : n : p$. Find the dimensions.*

Let a, b, c be the three dimensions.

Then $a : m = b : n = c : p$.

$\therefore an = bm$, and $bp = cn$. § 327

$\therefore a = \frac{bm}{n}$, and $c = \frac{bp}{n}$.

Now $V = a \times b \times c$. § 622

$\therefore V = \frac{bm}{n} \times b \times \frac{bp}{n} = \frac{b^3mp}{n^2}$.

$\therefore b = \sqrt[3]{\frac{n^2V}{mp}}$.

$\therefore a = \sqrt[3]{\frac{m^2V}{np}}$, and $c = \sqrt[3]{\frac{p^2V}{mn}}$.

Ex. 657. *Find the lateral area of a regular pyramid, if the slant height is 16 feet and the base is a regular hexagon with side 12 feet.*

$S = \frac{1}{2}L \times P = \frac{1}{2} \times 16 \times (6 \times 12) = 576$. § 643

576 sq. ft. *Ans.*

Ex. 658. *Find the lateral area of a regular pyramid, if the slant height is 8 feet and the base is a regular pentagon with side 5 feet.*

$S = \frac{1}{2}L \times P = \frac{1}{2} \times 8 \times (5 \times 5) = 100$. § 643

100 sq. ft. *Ans.*

Ex. 659. *Find the total surface of a regular pyramid, if the slant height is 6 feet and the base is a square with side 4 feet.*

$S = \frac{1}{2} L \times P = \frac{1}{2} \times 6 \times (4 \times 4) = 48.$ § 643

$B = 4 \times 4 = 16.$ § 398

$T = 48$ sq. ft. $+ 16$ sq. ft. $= 64$ sq. ft.

Ex. 660. *The slant height of a regular pyramid is divided in the ratio 1 : 3 by a plane parallel to the base. Find the ratio of the base to the section.*

The plane ∥ to the base divides the altitude in the ratio 1 : 3. § 645

If the upper segment of the altitude is the larger,

base : section $= 4^2 : 3^2 = 16 : 9.$ § 646

If the lower segment of the altitude is the larger,

base : section $= 4^2 : 1^2 = 16 : 1.$ § 646

Ex. 661. *The section of a pyramid made by a plane parallel to the base is half as large as the base. Find the ratio of the segments into which the altitude is divided by the plane.*

Let $x =$ the distance of the plane from the vertex.

Then $H - x =$ the distance of the plane from the base.

$\therefore$ base : section $= 2 : 1 = H^2 : x^2.$ § 646

$\therefore H : x = \sqrt{2} : \sqrt{1} = 1.414 : 1.$ § 338

$\therefore H - x : x = 1.414 - 1 : 1 = 0.414 : 1.$ § 333

Ex. 662. *Find the volume in cubic feet of a regular pyramid when its base is a square, each side measuring 3 feet 4 inches, and its height is 9 feet.*

3 ft. 4 in. $= 3\frac{1}{3}$ ft.

$V = \frac{1}{3} B \times H = \frac{1}{3} \times (3\frac{1}{3} \times 3\frac{1}{3} \times 9)$ cu. ft. $= 33\frac{1}{3}$ cu. ft. § 652

Ex. 663. *Find the volume in cubic feet of a regular pyramid when its base is an equilateral triangle, each side measuring 10 feet, and its height is 15 feet.*

$B = \frac{10^2\sqrt{3}}{4}$ sq. ft. $= 25\sqrt{3}$ sq. ft. Ex. 404

$V = \frac{1}{3} \times (15 \times 25\sqrt{3})$ cu ft. $= 125\sqrt{3}$ cu. ft. $= 216.5$ cu. ft. § 652

Ex. 664. *Find the volume in cubic feet of a regular pyramid when its base is a regular hexagon, each side measuring 4 feet, and its height is 20 feet.*

$r = \frac{1}{2} R\sqrt{3} = \frac{1}{2} \times 4\sqrt{3} = 2\sqrt{3}.$ Ex. 442

$B = \frac{1}{2} r \times P = \frac{1}{2} \times 2\sqrt{3} \times 6 \times 4 = 24\sqrt{3}.$ § 459

$V = \frac{1}{3} B \times H = \frac{1}{3} \times (24\sqrt{3} \times 20) = 160\sqrt{3} = 277.1.$ § 652

277.1 cu. ft. *Ans.*

Ex. 665. *Find the total surface in square feet of a regular pyramid when each side of its square base is* 10 *feet, and the slant height is* 18 *feet.*

$S = \frac{1}{2} L \times P = \frac{1}{2} \times 18 \times (4 \times 10) = 360.$ § 643

$B = 10 \times 10 = 100.$ § 398

$T = 360$ sq. ft. $+ 100$ sq. ft. $= 460$ sq. ft.

Ex. 666. *Find the total surface in square feet of a regular pyramid when each side of its triangular base is* 8 *feet, and the slant height is* 16 *feet.*

$S = \frac{1}{2} L \times P = \frac{1}{2} \times 16 \times (3 \times 8) = 192.$ § 643

$B = \frac{1}{4} a^2\sqrt{3} = \frac{1}{4} \times 8^2 \times \sqrt{3} = 16\sqrt{3} = 27.7.$ Ex. 404

$T = 192$ sq. ft. $+ 27.7$ sq. ft. $= 219.7$ sq. ft.

Ex. 667. *Find the total surface in square feet of a regular pyramid when each side of its square base is* 32 *feet, and the perpendicular height is* 72 *feet.*

$L = \sqrt{72^2 + 16^2} = \sqrt{5440} = 73.75.$ § 371

$S = \frac{1}{2} L \times P = \frac{1}{2} \times 73.75 \times 4 \times 32 = 4720.$ § 643

$B = 32 \times 32 = 1024.$ § 398

$T = 4720$ sq. ft. $+ 1024$ sq. ft. $= 5744$ sq. ft.

Ex. 668. *Find the height in feet of a regular pyramid when the volume is* 26 *cubic feet* 936 *cubic inches, and each side of the square base is* 3 *feet* 6 *inches.*

26 cu. ft. 936 cu. in. $= 26\frac{13}{24}$ cu. ft.

3 ft. 6 in. $= 3\frac{1}{2}$ ft.

$V = \frac{1}{3} H \times 3\frac{1}{2} \times 3\frac{1}{2}.$ § 652

$26\frac{13}{24} = \frac{1}{3} H \times \frac{7}{2} \times \frac{7}{2}.$

$\therefore H = \dfrac{26\frac{13}{24}}{\frac{1}{3} \times \frac{7}{2} \times \frac{7}{2}}$ ft. $= 6\frac{1}{2}$ ft.

Ex. 669. *Find the height in feet of a pyramid when the volume is* 20 *cubic feet, and the sides of its triangular base are* 5 *feet,* 4 *feet, and* 3 *feet.*

$B = \sqrt{6 \times 1 \times 2 \times 3} = 6.$ Ex. 405

$V = \frac{1}{3} B \times H.$ § 652

$20 = \frac{1}{3} \times 6\, H.$

$\therefore H = \dfrac{20}{\frac{1}{3} \times 6}$ ft. $= 10$ ft.

Ex. 670. *The base edge of a regular pyramid with a square base measures* 40 *feet, the lateral edge* 101 *feet. Find its volume in cubic feet.*

Diagonal of base $= \sqrt{40^2 + 40^2} = 40\sqrt{2}$. § 371

$\frac{1}{2}$ of $40\sqrt{2}$ ft. $= 20\sqrt{2}$ ft.

$H = \sqrt{101^2 - (20\sqrt{2})^2} = \sqrt{10201 - 800} = \sqrt{9401} = 96.9587$. § 372

$V = \frac{1}{3} \times 40^2 \times 96.9587 = 51{,}711.31$. § 652

51,711.31 cu. ft. *Ans.*

Ex. 671. *Find the volume of a regular pyramid whose slant height is 12 feet, and whose base is an equilateral triangle inscribed in a circle that has a radius of 10 feet.*

Let $2x$ denote a side of the triangle.

The altitude of the triangular base is 15 ft. Ex. 27

Then $4x^2 = x^2 + 15^2$.

$3x^2 = 225$.

$x^2 = 75$.

$x = 8.660$.

$2x = 17.320$.

$H = \sqrt{12^2 - 5^2} = \sqrt{144 - 25} = \sqrt{119} = 10.908$.

$V = \frac{1}{3} \times \frac{1}{2}(15 \times 17.320) \times 10.908 = 472.3164$. § 652

472.32 cu. ft. *Ans.*

Ex. 672. *Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.*

$$L = \sqrt{\tfrac{1}{4}a^2 + H^2}.$$

$$T = 2aL + a^2.$$

$$\therefore T = 2a\sqrt{\tfrac{1}{4}a^2 + H^2} + a^2.$$

$$\sqrt{\tfrac{1}{4}a^2 + H^2} = \frac{T - a^2}{2a}.$$

$$\tfrac{1}{4}a^2 + H^2 = \frac{T^2 - 2a^2T + a^4}{4a^2}.$$

$$H^2 = \frac{T^2 - 2a^2T}{4a^2}.$$

$$\therefore H = \frac{1}{2a}\sqrt{T(T - 2a^2)}.$$

$$V = \tfrac{1}{3}a^2H = \tfrac{1}{3}a^2 \times \frac{1}{2a}\sqrt{T(T - 2a^2)} = \frac{a}{6}\sqrt{T(T - 2a^2)}. \quad \text{§ 652}$$

Ex. 673. *The base edge of a regular pyramid whose base is a square is a, the total surface T. Find the height of the pyramid.*

$$H = \frac{1}{2a}\sqrt{T(T - 2a^2)}. \quad \text{Ex. 672}$$

Ex. 674. *The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T. Find the length of one edge.*

$$L = \sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3\,a^2}{4}} = \frac{a}{2}\sqrt{3}.$$

$$T = 2\,aL + a^2.$$

$$\therefore T = a^2\sqrt{3} + a^2.$$

$$a^2 = \frac{T}{\sqrt{3}+1}.$$

$$\therefore a = \sqrt{\frac{T}{\sqrt{3}+1}} = \sqrt{\frac{T}{2}(\sqrt{3}-1)}.$$

Ex. 675. *Find the base edge a of a regular pyramid with a square base, having given the height H and the total surface T.*

$$L = \sqrt{\frac{a^2}{4} + H^2}.$$

$$T = 2\,aL + a^2.$$

$$\therefore T = 2\,a\sqrt{\frac{a^2}{4} + H^2} + a^2.$$

$$\sqrt{\frac{a^2}{4} + H^2} = \frac{T - a^2}{2\,a}.$$

$$\frac{a^2}{4} + H^2 = \frac{T^2 - 2\,a^2T + a^4}{4\,a^2}.$$

$$a^4 + 4\,a^2H^2 = T^2 - 2\,a^2T + a^4.$$

$$a^2\,(2\,T + 4\,H^2) = T^2.$$

$$a^2 = \frac{T^2}{2\,T + 4\,H^2} = \frac{2\,T^2}{4\,(T + 2\,H^2)}.$$

$$\therefore a = \frac{T}{2}\sqrt{\frac{2}{T + 2\,H^2}}.$$

Ex. 676. *The homologous edges of two similar tetrahedrons are as 6 : 7. Find the ratio of their surfaces and of their volumes.*

$$S : S' = 6^2 : 7^2 = 36 : 49. \qquad \S\ 670$$

$$V : V' = 6^3 : 7^3 = 216 : 343. \qquad \S\ 671$$

Ex. 677. *If the edge of a tetrahedron is a, find the homologous edge of a similar tetrahedron twice as large.*

$$V : V' = a^3 : 2\,a^3. \qquad \S\ 671$$

$$\text{Edge of } V' = a\sqrt[3]{2}.$$

Ex. 678. *The diameter of a well is 6 feet, and the water is 7 feet deep. How many gallons of water are there in the well, reckoning 7½ gallons to the cubic foot?*

$V = \pi R^2 H$ § 700

$= 3.1416 \times (3^2 \times 7)$ cu. ft.

$= 3.1416 \times 3^2 \times 7 \times 7.5$ gal. $= 1484.4$ gal.

Ex. 679. *When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 40 centimeters. Find the volume of the body.*

$V = \pi R^2 H = 3.1416 \times (30^2 \times 40)^{\text{ccm}} = 113{,}097.6^{\text{ccm}}$. § 700

Ex. 680. *How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semicircle with a radius of 18 feet?*

18 ft. $= 6$ yd.

$V = \frac{1}{2}\pi R^2 H = \frac{1}{2} \times 3.1416 \times (6^2 \times 100)$ cu. yd. $= 5654.88$ cu. yd. § 700

Ex. 681. *How many square feet of sheet iron are required to make a funnel 18 inches in diameter and 40 feet long?*

18 in. $= 1\frac{1}{2}$ ft.

$S = 2\pi RH = 3.1416 \times (1\frac{1}{2} \times 40)$ sq. ft. $= 188.5$ sq. ft. § 698

Ex. 682. *Find the radius of a cylindrical pail 14 inches high that will hold exactly 2 cubic feet.*

2 cu. ft. $= 2 \times 1728$ cu. in. $= 3456$ cu. in.

$V = \pi R^2 H$. § 700

$3456 = \pi R^2 \times 14$.

$R^2 = \dfrac{3456}{14\,\pi} = 246\frac{6}{7} \times 0.31831 = 78.5771$.

$\therefore R = \sqrt{78.5771}$ in. $= 8.86$ in.

Ex. 683. *The height of a cylindrical vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.*

$20^{\text{l}} = 20{,}000^{\text{ccm}}$.

$V = \pi R^2 H$. § 700

$20000 = \pi R^2 \times 2R = 2\pi R^3$.

$\therefore R^3 = \dfrac{20000}{2\,\pi} = 10000 \times 0.31831 = 3183.1$.

$\therefore R = \sqrt[3]{3183.1^{\text{cm}}} = 14.71^{\text{cm}}$.

$H = 2R = 2 \times 14.71^{\text{cm}} = 29.42^{\text{cm}}$.

Ex. 684. *If the total surface of a right circular cylinder is T, and the radius of the base is R, find the height of the cylinder.*

$$T = 2\pi RH + 2\pi R^2. \qquad \S\ 698$$

$$2\pi RH = T - 2\pi R^2.$$

$$\therefore H = \frac{T}{2\pi R} - R.$$

Ex. 685. *If the lateral surface of a right circular cylinder is S, and the volume is V, find the radius of the base and the height.*

$$S = 2\pi RH. \qquad \S\ 698$$

$$V = \pi R^2 H. \qquad \S\ 700$$

$$\overline{SR = 2\pi R^2 H.}$$

$$2V = 2\pi R^2 H.$$

$$\therefore SR = 2V.$$

$$\therefore R = \frac{2V}{S}.$$

$$\therefore S = 2\pi \times \frac{2V}{S} \times H.$$

$$\therefore H = \frac{S}{2\pi \times \frac{2V}{S}} = \frac{S^2}{4\pi V}.$$

Ex. 686. *If the circumference of the base of a right circular cylinder is C, and the height H, find the volume V.*

$$R = \frac{C}{2\pi}. \qquad \S\ 458$$

$$\therefore V = \pi R^2 H = \pi \left(\frac{C}{2\pi}\right)^2 H = \frac{C^2 H}{4\pi}. \qquad \S\ 700$$

Ex. 687. *Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.*

$$H = 2R.$$

$$V = \pi R^2 H = 2\pi R^3. \qquad \S\ 700$$

$$T = 2\pi RH + 2\pi R^2 \qquad \S\ 698$$

$$= 4\pi R^2 + 2\pi R^2 = 6\pi R^2.$$

$$\therefore R^2 = \frac{T}{6\pi}.$$

$$\therefore R = \sqrt{\frac{T}{6\pi}}.$$

$$\therefore R^3 = \frac{T}{6\pi}\sqrt{\frac{T}{6\pi}}.$$

$$\therefore V = 2\pi \times \frac{T}{6\pi}\sqrt{\frac{T}{6\pi}} = \frac{T}{3}\sqrt{\frac{T}{6\pi}} = \frac{T}{18\pi}\sqrt{6\pi T}.$$

Ex. 688. *If the circumference of the base of a right circular cylinder is C, and the total surface is T, find the volume V.*

$$V = \pi R^2 H. \qquad \S\ 700$$

$$T = 2\pi R^2 + 2\pi RH. \qquad \S\ 698$$

$$C = 2\pi R. \qquad \S\ 458$$

$$\therefore R = \frac{C}{2\pi}.$$

$$\therefore V = \pi \times \left(\frac{C}{2\pi}\right)^2 \times H = \frac{C^2 H}{4\pi},$$

and

$$T = 2\pi \times \left(\frac{C}{2\pi}\right)^2 + 2\pi \times \frac{C}{2\pi} \times H$$

$$= \frac{C^2}{2\pi} + CH.$$

$$\therefore CH = T - \frac{C^2}{2\pi}.$$

$$\therefore \frac{C^2 H}{4\pi} = \frac{CT}{4\pi} - \frac{C^3}{8\pi^2}.$$

$$\therefore V = \frac{CT}{4\pi} - \frac{C^3}{8\pi^2}.$$

$$\therefore V = \frac{C}{8\pi^2}(2\pi T - C^2).$$

Ex. 689. *If the volume of a right circular cylinder is V, and the altitude is H, find the total surface T.*

$$V = \pi R^2 H. \qquad \S\ 700$$

$$\therefore R^2 = \frac{V}{\pi H} \text{ and } R = \sqrt{\frac{V}{\pi H}}.$$

$$T = 2\pi RH + 2\pi R^2 \qquad \S\ 698$$

$$= 2\pi \sqrt{\frac{V}{\pi H}} \times H + 2\pi \times \frac{V}{\pi H}$$

$$= 2\pi H \sqrt{\frac{V}{\pi H}} + \frac{2V}{H}$$

$$= 2\sqrt{\pi V H} + \frac{2V}{H}.$$

Ex. 690. *If V is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude H and the total surface T.*

$$H = 2R.$$

$$\therefore R = \frac{H}{2}.$$

$$V = \pi R^2 H = \pi \left(\frac{H}{2}\right)^2 \times H = \frac{\pi H^3}{4}. \qquad \S\ 700$$

$$\therefore H^3 = \frac{4V}{\pi}.$$

$$\therefore H = \sqrt[3]{\frac{4V}{\pi}}.$$

$$T = 2\pi RH + 2\pi R^2 \qquad \S\ 698$$

$$= 2\pi \times \frac{H}{2} \times H + 2\pi\left(\frac{H}{2}\right)$$

$$= \pi H^2 + \tfrac{1}{2}\pi H^2$$

$$= \tfrac{3}{2}\pi H^2$$

$$= \tfrac{3}{2}\pi\left(\sqrt[3]{\frac{4V}{\pi}}\right)^2$$

$$= \tfrac{3}{2}\pi \sqrt[3]{\frac{16V^2}{\pi^2}}$$

$$= 3\sqrt[3]{2\pi V^2}.$$

Ex. 691. *If T is the total surface, and H the altitude of a right circular cylinder, find the radius R and the volume V.*

$$T = 2\pi RH + 2\pi R^2. \qquad \S\ 698$$

$$2\pi R^2 + 2\pi RH = T.$$

$$4\pi^2 R^2 + 4\pi^2 RH + \pi^2 H^2 = 2\pi T + \pi^2 H^2.$$

$$2\pi R + \pi H = \sqrt{2\pi T + \pi^2 H^2}.$$

$$2\pi R = -\pi H + \sqrt{2\pi T + \pi^2 H^2}.$$

$$R = -\frac{H}{2} + \frac{1}{2\pi}\sqrt{2\pi T + \pi^2 H^2}.$$

$$V = \pi R^2 H \qquad \S\ 700$$

$$= \pi\left(-\frac{H}{2} + \frac{1}{2\pi}\sqrt{2\pi T + \pi^2 H^2}\right)^2 \times H$$

$$= \pi\left(\frac{H^2}{4} - \frac{H}{2\pi}\sqrt{2\pi T + \pi^2 H^2} + \frac{T}{2\pi} + \frac{H^2}{4}\right) \times H$$

$$= \pi H\left(\frac{H^2}{2} - \frac{H}{2\pi}\sqrt{2\pi T + \pi^2 H^2} + \frac{T}{2\pi}\right)$$

$$= \frac{H}{2}(\pi H^2 - H\sqrt{2\pi T + \pi^2 H^2} + T).$$

Ex. 692. *The radii of the bases of the frustum of a right circular cone are 20 inches and 13 inches, respectively. If the altitude of the frustum is 15 inches, and is bisected by a plane parallel to the bases, what is the lateral area of each frustum made by the plane?*

$L = \sqrt{15^2 + 7^2} = \sqrt{274} = 16.552.$ § 371

The slant height of each frustum $= \frac{1}{2}$ of $16.552 = 8.276.$

The radius of the mid-section $= \frac{1}{2}(20 + 13) = 16.5.$ §§ 187, 189

The lateral area of the lower frustum $= \frac{1}{2}(C + c) \times L$ § 727

$= \frac{1}{2}(2\pi R + 2\pi r) \times L$

$= \pi L(R + r)$

$= 3.1416 \times 8.276 \times (20 + 16.5)$

$= 948.9956784.$

The lateral area of the upper frustum $= 3.1416 \times 8.276 \times (16.5 + 13)$

$= 766.9965072.$

949 sq. in.; 767 sq. in. *Ans.*

Ex. 693. *The radius of the base of a right circular cone is* 8 *feet, and the altitude is* 10 *feet. Find the area of its lateral surface, the area of its total surface, and the volume.*

$L = \sqrt{10^2 + 8^2} = \sqrt{164} = 12.806.$ § 371

$S = \pi RL = 3.1416 \times (8 \times 12.806)$ sq. ft. $= 321.85$ sq. ft. § 723

$B = \pi R^2 = 3.1416 \times (8^2)$ sq. ft. $= 201.06$ sq. ft. § 463

$T = 321.85$ sq. ft. $+ 201.06$ sq. ft. $= 522.91$ sq. ft.

$V = \frac{1}{3}\pi R^2 H = \frac{1}{3} \times 3.1416 \times (8^2 \times 10)$ cu. ft. $= 670.21$ cu. ft. § 725

Ex. 694. *The height of a right circular cone is equal to the diameter of its base. Find the ratio of the area of the base to the area of the lateral surface.*

$B = \pi R^2 = \frac{1}{4}\pi H^2.$ § 463

$L = \sqrt{H^2 + \frac{H^2}{4}} = \sqrt{\frac{5H^2}{4}} = \frac{1}{2}H\sqrt{5}.$ § 371

$S = \frac{1}{2}\pi H \times \frac{1}{2}H\sqrt{5} = \frac{1}{4}\pi H^2\sqrt{5}.$ § 723

$\therefore \frac{B}{S} = \frac{\frac{1}{4}\pi H^2}{\frac{1}{4}\pi H^2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5}.$

Ex. 695. *The slant height of a right circular cone is* 2 *feet. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts?*

$S = \pi RL = 2\pi R.$ § 723

$S : S' = 2 : 1.$

$\therefore L : L' = \sqrt{2} : 1.$

$2 : L' = \sqrt{2} : 1.$

$\therefore L' = \sqrt{2} = 1.414.$ 1.414 ft. *Ans.*

Ex. 696. *What does the volume V of a circular cone become, if the altitude is doubled? If the radius of the base is doubled? If both the altitude and the radius of the base are doubled?*

$$V = \tfrac{1}{3}\pi R^2 H. \qquad \S\ 725$$

If H is doubled, V is multiplied by 2.

If R is doubled, V is multiplied by 2^2 or 4.

If H and R are both doubled, V is multiplied by 2×4 or 8.

Ex. 697. *The slant height L of a right circular cone is equal to the diameter of the base. Find the total surface T.*

$$2R = L.$$
$$\therefore R = \tfrac{1}{2}L.$$
$$T = \pi RL + \pi R^2 \qquad \S\ 723$$
$$= \pi(\tfrac{1}{2}L)L + \pi(\tfrac{1}{2}L)^2$$
$$= \tfrac{1}{2}\pi L^2 + \tfrac{1}{4}\pi L^2 = \tfrac{3}{4}\pi L^2.$$

Ex. 698. *If T is the total surface of a right circular cone whose slant height equals the diameter of the base, find the volume V.*

$$2R = L.$$
$$T = \pi RL + \pi R^2 \qquad \S\ 723$$
$$= \pi R(2R) + \pi R^2$$
$$= 3\pi R^2.$$
$$\therefore R^2 = \frac{T}{3\pi}.$$
$$\therefore R = \sqrt{\frac{T}{3\pi}}.$$
$$L^2 = H^2 + R^2. \qquad \S\ 371$$
$$(2R)^2 = H^2 + R^2.$$
$$\therefore H^2 = 3R^2.$$
$$\therefore H = R\sqrt{3}$$
$$= \sqrt{\frac{T}{\pi}}.$$
$$V = \tfrac{1}{3}\pi R^2 H \qquad \S\ 725$$
$$= \tfrac{1}{3}\pi\left(\frac{T}{3\pi}\right)\sqrt{\frac{T}{\pi}} = \frac{T}{9}\sqrt{\frac{T}{\pi}}.$$

Ex. 699. *If T is the total surface of a right circular cone, and R is the radius of the base, find the volume V.*

$$T = \pi RL + \pi R^2. \qquad \S\ 723$$

But
$$L^2 = H^2 + R^2. \qquad \S\ 371$$

$$\therefore L = \sqrt{H^2 + R^2}.$$

$$\therefore T = \pi R\sqrt{H^2 + R^2} + \pi R^2.$$

$$\sqrt{H^2 + R^2} = \frac{T - \pi R^2}{\pi R}.$$

$$H^2 + R^2 = \frac{T^2 - 2\pi R^2 T + \pi^2 R^4}{\pi^2 R^2}.$$

$$H^2 = \frac{T^2 - 2\pi R^2 T}{\pi^2 R^2}.$$

$$H = \frac{1}{\pi R}\sqrt{T(T - 2\pi R^2)}.$$

But $V = \frac{1}{3}\pi R^2 H$ § 725

$$= \tfrac{1}{3}\pi R^2 \times \frac{1}{\pi R}\sqrt{T(T - 2\pi R^2)}$$

$$= \frac{R}{3}\sqrt{T(T - 2\pi R^2)}.$$

Ex. 700. *If T is the total surface of a right circular cone, and S is the lateral surface, find the volume V.*

$$T = \pi RL + \pi R^2 = \pi R(L + R), \quad \text{§ 723}$$

and $S = \pi RL.$ § 723

But $L^2 = H^2 + R^2.$ § 371

$$\therefore H = \sqrt{L^2 - R^2}.$$

$$V = \tfrac{1}{3}\pi R^2 H = \tfrac{1}{3}\pi R^2\sqrt{L^2 - R^2} \quad \text{§ 725}$$

$$= \tfrac{1}{3}\pi R^2\sqrt{(L + R)(L - R)}$$

$$= \tfrac{1}{3}\sqrt{R(L + R)(\pi R^2)(\pi R)(L - R)}$$

$$= \frac{1}{3}\sqrt{\frac{[\pi R(L + R)][\pi R^2][\pi R(L - R)]}{\pi}}$$

$$= \frac{1}{3}\sqrt{\frac{T(T - S)(2S - T)}{\pi}}.$$

Ex. 701. *Show that the prismatoid formula can be used for finding the volume of the frustum of a cone, for finding the volume of a cone, for finding the volume of a cylinder.*

The formula for the volume of the frustum of a cone is

$$V = \tfrac{1}{3}H(B + b + \sqrt{B \times b}). \quad \text{§ 729}$$

If we inscribe the frustum of a pyramid in the frustum of the cone, and increase the number of its sides indefinitely, by the Method of the Theory of Limits, we have

$$\tfrac{1}{3}H(B + b + 4M) = \tfrac{1}{3}H(B + b + \sqrt{B \times b}). \quad \text{§ 734}$$

By the same method, inscribing in the cone a pyramid with base a regular polygon, and inscribing in the cylinder a prism with bases regular polygons, we have $V = \frac{1}{3}\pi R^2 H$, § 725

and $V = \pi R^2 H$. § 700

Ex. 702. *How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are* 28 *inches and* 14 *inches, respectively, and the height is* 24 *inches?*

$$L = \sqrt{24^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25. \quad \S\ 371$$

$$S = \pi L (R + R') \quad \S\ 727$$

$$= 3.1416 \times 25 \times (14 + 7) = 1649.34.$$

1649.34 sq. in. = 11 sq. ft. 65.34 sq. in. *Ans.*

Ex. 703. *Find the expense, at* 60 *cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is* 12 *feet, and the radii of whose bases are* 3 *feet* 6 *inches and* 2 *feet* 4 *inches, respectively.*

$$S = \pi L (R + R') = 3.1416 \times 12 \times (3\tfrac{1}{2} + 2\tfrac{1}{3}) \quad \S\ 727$$

$$= 219.912.$$

$$\text{Cost} = 219.912 \times \$0.60 = \$131.95.$$

Ex. 704. *The slant height of the frustum of a regular pyramid is* 20 *feet; the sides of its square bases* 40 *feet and* 16 *feet. Find the volume.*

$$H = \sqrt{20^2 - 12^2} = \sqrt{400 - 144} = \sqrt{256} = 16. \quad \S\ 372$$

$$V = \tfrac{1}{3} H (B + b + \sqrt{Bb}) = \tfrac{1}{3} H (40^2 + 16^2 + 40 \times 16) \quad \S\ 657$$

$$= \tfrac{1}{3} H (1600 + 256 + 640)$$

$$= \tfrac{1}{3} \times 16 \times 2496$$

$$= 13{,}312. \qquad 13{,}312 \text{ cu. ft. } \textit{Ans.}$$

Ex. 705. *If the bases of the frustum of a pyramid are regular hexagons whose sides are* 1 *foot and* 2 *feet, respectively, and the volume of the frustum is* 12 *cubic feet, find the altitude.*

$$\sqrt{1^2 - (\tfrac{1}{2})^2} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3} = 0.866.$$

$$b = 6 \times \tfrac{1}{2}(1 \times 0.866) = 2.598. \quad \S\ 459$$

$$b : B = 1^2 : 2^2 = 1 : 4. \quad \S\ 446$$

$$\therefore B = 4 \times 2.598 = 10.392.$$

$$V = \tfrac{1}{3} H (B + b + \sqrt{Bb}). \quad \S\ 657$$

$$12 = \tfrac{1}{3} H (10.392 + 2.598 + 5.196).$$

$$36 = 18.186\, H.$$

$$H = \frac{36}{18.186} = 1.98. \qquad 1.98 \text{ ft. } \textit{Ans.}$$

Ex. 706. *The frustum of a right circular cone* 14 *feet high has a volume of* 924 *cubic feet. Find the radii of its bases if their sum is* 9 *feet.*

Let r and $9 - r$ be the radii.

$V = \frac{1}{3}\pi H(R^2 + r^2 + Rr)$. § 730

$924 = \frac{1}{3} \times \frac{22}{7} \times 14\,[R^2 + (9 - R)^2 + R(9 - R)]$

$= \frac{44}{3}(R^2 + 81 - 18R + R^2 + 9R - R^2)$.

$63 = R^2 - 9R + 81$.

$R^2 - 9R = -18$.

$4R^2 - (\ ) + 81 = 9$.

$2R - 9 = \pm 3$.

$2R = 12$ or 6.

$R = 6$ or 3.

$9 - R = 3$ or 6.

Therefore, the radii are 6 ft. and 3 ft.

Ex. 707. *From a right circular cone whose slant height is* 30 *feet, and the circumference of whose base is* 10 *feet, there is cut off by a plane parallel to the base a cone whose slant height is* 6 *feet. Find the lateral area and the volume of the frustum.*

If the circumference of the section is denoted by C',

$C' = \frac{6}{30}$ of $10 = 2$.

Then $S = \frac{1}{2} \times 24\,(10 + 2) = 12 \times 12 = 144$. § 727

Now $2\pi R = 10$, and $2\pi r = 2$.

$\therefore R = \frac{5}{\pi}$, and $r = \frac{1}{\pi}$.

$H = \sqrt{24^2 - \left(\frac{4}{\pi}\right)^2} = \sqrt{576 - \frac{16}{\pi^2}} = 23.966$.

$V = \frac{1}{3}\pi H(R^2 + r^2 + Rr)$ § 730

$= \frac{1}{3}\pi \times 23.966\left(\frac{25}{\pi^2} + \frac{1}{\pi^2} + \frac{5}{\pi^2}\right)$

$= \frac{1}{3} \times 23.966 \times \frac{31}{\pi}$

$= \frac{1}{3} \times 23.966 \times 31 \times 0.31831$

$= 78.829$. 78.83 cu. ft. *Ans.*

Ex. 708. *Find the difference between the volume of the frustum of a pyramid whose bases are squares,* 8 *feet and* 6 *feet, respectively, on a side and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.*

$V = \frac{1}{3}H(B + b + \sqrt{Bb}) = \frac{1}{3}H(64 + 36 + 48) = \frac{1}{3}H \times 148 = 49\frac{1}{3}H$. § 657

$V' = 49H$. $\therefore V - V' = \frac{1}{3}H$ cu. ft.

Ex. 709. *A Dutch windmill in the shape of the frustum of a right cone is* 12 *meters high. The outer diameters at the bottom and the top are* 16 *meters and* 12 *meters, the inner diameters* 12 *meters and* 10 *meters. How many cubic meters of stone were required to build it?*

$V = \frac{1}{3}\pi H(R^2 + r^2 + Rr) = \frac{1}{3}\pi \times 12(64 + 36 + 48) = 4\pi \times 148 = 592\pi.$

$V' = \frac{1}{3}\pi \times 12(36 + 25 + 30) = 4\pi \times 91 = 364\pi.$

$592\pi - 364\pi = 228\pi = 716.2848.$

716.28$^{\text{cbm}}$. *Ans.*

Ex. 710. *The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is* 180 *feet, and its upper and lower bases are squares whose sides are* 10 *feet and* 16 *feet, respectively. The flue is throughout a square whose side is* 7 *feet. How many cubic feet of material does the chimney contain?*

$V = \frac{1}{3}H(B + b + \sqrt{Bb}) = \frac{1}{3} \times 180(256 + 100 + 160)$ § 657

$= 60 \times 516 = 30{,}960.$

$V' = 180 \times 7^2 = 180 \times 49 = 8820.$ § 624

30,960 cu. ft. − 8820 cu. ft. = 22,140 cu. ft.

Ex. 711. *Find the volume V of the frustum of a cone of revolution, having given the slant height L, the height H, and the lateral area S.*

$S = \pi LR + \pi Lr.$ § 727

$r = R - \sqrt{L^2 - H^2}.$

$S = \pi LR + \pi LR - \pi L\sqrt{L^2 - H^2}.$

$S = 2\pi LR - \pi L\sqrt{L^2 - H^2}.$

$\therefore R = \frac{S}{2\pi L} + \frac{1}{2}\sqrt{L^2 - H^2},$

and $r = \frac{S}{2\pi L} - \frac{1}{2}\sqrt{L^2 - H^2}.$

$V = \frac{1}{3}\pi H(R^2 + r^2 + Rr)$ § 730

$= \frac{1}{3}\pi H\left[\frac{S^2}{4\pi^2L^2} + \frac{S\sqrt{L^2 - H^2}}{2\pi L} + \frac{1}{4}(L^2 - H^2)\right.$

$+ \frac{S^2}{4\pi^2L^2} - \frac{S\sqrt{L^2 - H^2}}{2\pi L} + \frac{1}{4}(L^2 - H^2)$

$\left. + \frac{S^2}{4\pi^2L^2} - \frac{1}{4}(L^2 - H^2)\right]$

$= \frac{1}{3}\pi H\left[\frac{3S^2}{4\pi^2L^2} + \frac{1}{4}(L^2 - H^2)\right].$

Ex. 712. *A cube each edge of which is* 12 *inches is transformed into a right prism whose base is a rectangle* 16 *inches long and* 12 *inches wide. Find the height of the prism, and the difference between its total area and the total area of the cube.*

$V = (12^3)$ cu. in. $= 1728$ cu. in. § 623

$B = (16 \times 12)$ sq. in. $= 192$ sq. in. § 398

$H = (1728 \div 192)$ in. $= 9$ in.

$T = 6 \times (12 \times 12)$ sq. in. $= 864$ sq. in.

$T' = 2(16 \times 12 + 16 \times 9 + 12 \times 9)$ sq. in.

$= 2(192 + 144 + 108)$ sq. in.

$= 2 \times 444$ sq. in. $= 888$ sq. in.

$T' - T = 888$ sq. in. $- 864$ sq. in. $= 24$ sq. in.

Ex. 713. *The dimensions of a rectangular parallelopiped are* a, b, c. *Find* (*i*) *the height of an equivalent right circular cylinder having* a *for the radius of its base;* (*ii*) *the height of an equivalent right circular cone having* a *for the radius of its base.*

(*i*) $V = abc = \pi a^2 H.$ §§ 622, 700

$\therefore H = \dfrac{abc}{\pi a^2} = \dfrac{bc}{\pi a}.$

(*ii*) $V = abc = \frac{1}{3}\pi R^2 H.$ § 725

$\therefore H = \dfrac{3abc}{\pi a^2} = \dfrac{3bc}{\pi a}.$

Ex. 714. *A regular pyramid* 12 *feet high is transformed into a regular prism with an equivalent base. Find the height of the prism.*

$V = \frac{1}{3}BH = \frac{1}{3}B \times 12.$ § 652

$V' = BH.$ § 628

$\frac{1}{3}B \times 12 = BH.$

$H = \dfrac{4B}{B} = 4.$ 4 ft. *Ans.*

Ex. 715. *The diameter of a cylinder is* 14 *feet, and its height is* 8 *feet. Find the height of an equivalent right prism the base of which is a square with a side* 4 *feet long.*

$V = \pi R^2 H$ § 700

$= 3.1416 \times 7^2 \times 8 = 1231.51.$

$V' = BH.$ § 628

$1231.51 = 4^2 H = 16H.$

$\therefore H = \dfrac{1231.51}{16} = 76.97.$ 76.97 ft. *Ans.*

Ex. 716. *If one edge of a cube is a, what is the height H of an equivalent right circular cylinder whose radius is R?*

$$V = a^3. \qquad \S\ 623$$
$$V' = \pi R^2 H. \qquad \S\ 700$$
$$\therefore a^3 = \pi R^2 H.$$
$$\therefore H = \frac{a^3}{\pi R^2}$$

Ex. 717. *The heights of two equivalent right circular cylinders are in the ratio 4 : 9. If the diameter of the first is 6 feet, what is the diameter of the other?*

$$V = \pi R^2 H = \pi\, 3^2 H = 9\pi H. \qquad \S\ 700$$
$$V' = \pi R'^2 \left(\frac{9H}{4}\right) = \frac{9\pi R'^2 H}{4}.$$
$$\therefore \frac{9\pi R'^2 H}{4} = 9\pi H.$$
$$R'^2 = 4, \quad R' = 2.$$
$$\text{Diameter} = 2R' = 4 \text{ ft.}$$

Ex. 718. *A right circular cylinder 6 feet in diameter is equivalent to a right circular cone 7 feet in diameter. If the height of the cone is 8 feet, what is the height of the cylinder?*

$$V = \tfrac{1}{3}\pi R^2 H = \tfrac{1}{3}\pi\,(\tfrac{7}{2})^2 \times 8 = \frac{98\pi}{3}. \qquad \S\ 725$$
$$V' = \pi R'^2 H' = \pi\, 3^2 H' = 9\pi H'. \qquad \S\ 700$$
$$\therefore 9\pi H' = \frac{98\pi}{3}.$$
$$\therefore H' = \tfrac{98}{27} \text{ ft.} = 3.63 \text{ ft.}$$

Ex. 719. *The frustum of a regular pyramid 6 feet high has for bases squares 5 feet and 8 feet on a side. Find the height of an equivalent regular pyramid whose base is a square 12 feet on a side.*

$$V = \tfrac{1}{3} H(B + b + \sqrt{Bb}) \qquad \S\ 657$$
$$= \tfrac{1}{3} \times 6\,(64 + 25 + 40)$$
$$= 2 \times 129 = 258.$$
$$V' = \tfrac{1}{3} B'H'. \qquad \S\ 652$$
$$258 = \tfrac{1}{3} \times 144\, H'.$$
$$\therefore H' = 5\tfrac{3}{8} \text{ ft.}$$

Ex. 720. *The frustum of a cone of revolution is 5 feet high, and the diameters of its bases are 2 feet and 3 feet, respectively. Find the height*

of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases, and equidistant from the bases.

$$V = \tfrac{1}{3}\pi H(R^2 + r^2 + Rr) \qquad \S\ 730$$
$$= \tfrac{1}{3}\pi \times 5(\tfrac{9}{4} + 1 + \tfrac{3}{2})$$
$$= \tfrac{5}{3} \times \tfrac{19}{4} \times \pi$$
$$= \frac{95\,\pi}{12}.$$
$$V' = \pi R'^2 H'. \qquad \S\ 700$$
$$\frac{95\,\pi}{12} = \pi\, \tfrac{25}{16} H'.$$
$$\therefore H' = 5\tfrac{1}{15} \text{ ft.}$$

Ex. 721. *Find the edge of a cube equivalent to a regular tetrahedron whose edge measures 3 inches.*

Altitude of base $= \sqrt{9 - \tfrac{9}{4}} = \sqrt{\tfrac{27}{4}} = \tfrac{3}{2}\sqrt{3}$. § 372

$\tfrac{2}{3}$ of $\tfrac{3}{2}\sqrt{3} = \sqrt{3}$.

$H = \sqrt{3^2 - (\sqrt{3})^2} = \sqrt{9 - 3} = \sqrt{6}$. § 372

$B = \tfrac{1}{2} \times 3 \times \tfrac{3}{2}\sqrt{3} = \tfrac{9}{4}\sqrt{3}$. § 403

$V = \tfrac{1}{3} \times \tfrac{9}{4}\sqrt{3} \times \sqrt{6}$ § 652

$= \tfrac{9}{4}\sqrt{2} = \tfrac{9}{4} \times 1.41421 = 3.181973.$

An edge $= \sqrt[3]{3.181973}$ in. $= 1.47$ in.

Ex. 722. *Find the edge of a cube equivalent to a regular octahedron whose edge measures 3 inches.*

Let H denote the altitude of each of the two pyramids into which the octahedron can be divided, and V the volume of each pyramid.

Then $H = \sqrt{3^2 - (\tfrac{3}{2}\sqrt{2})^2} = \sqrt{9 - \tfrac{9}{2}} = \tfrac{3}{2}\sqrt{2}$, § 372

and $V = \tfrac{1}{3} \times 3^2 \times \tfrac{3}{2}\sqrt{2} = \tfrac{9}{2}\sqrt{2}$. § 652

$\therefore 2V = 9\sqrt{2} = 9 \times 1.41421 = 12.72789.$

An edge $= \sqrt[3]{12.72789}$ in. $= 2.33$ in.

Ex. 723. *The dimensions of a trunk are 4 feet, 3 feet, 2 feet. Find the dimensions of a trunk similar in shape that will hold four times as much.*

$1 : 4 = 4^3 : x^3$. § 672 $1 : 4 = 3^3 : x^3$. § 672 $1 : 4 = 2^3 : x^3$. § 672

$x^3 = 256$. $x^3 = 108$. $x^3 = 32$.

$x = 6.35$ ft. $x = 4.76$ ft. $x = 3.17$ ft.

Ex. 724. *By what number must the dimensions of a cylinder be multiplied to obtain a similar cylinder (i) whose surface shall be n times that of the first; (ii) whose volume shall be n times that of the first?*

(*i*) $\sqrt{n}$. § 701 (*ii*) $\sqrt[3]{n}$. § 701

Ex. 725. *A pyramid is cut by a plane parallel to the base which passes midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.*

$$V : V' = 2^3 : 1 = 8 : 1. \qquad \S\ 672$$

Ex. 726. *The height of a regular hexagonal pyramid is 36 feet, and one side of the base is 6 feet. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?*

$20 : 1 = 36^3 : H^3$. § 672 $\qquad 20 : 1 = 6^3 : a^3$. § 672

$$H^3 = \frac{36^3}{20} = 2332.8. \qquad a^3 = \frac{6^3}{20} = 10.8.$$

$$H = \sqrt[3]{2332.8}\text{ ft.} = 13.2625\text{ ft.} \qquad a = \sqrt[3]{10.8}\text{ ft.} = 2.2104\text{ ft.}$$

Ex. 727. *The length of one of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?*

$$V : V' = 4^3 : x^3. \qquad \S\ 672$$
$$2 : 1 = 64 : x^3.$$
$$\therefore x^3 = 32.$$
$$\therefore x = 3.17^{\text{m}}.$$

Ex. 728. *A lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base that divide the pyramid into three equivalent parts?*

$V : V' = a^3 : a'^3$. § 672 $\qquad V : V' = a^3 : a''^3$. § 672

$3 : 1 = a^3 : a'^3$. $\qquad 3 : 2 = a^3 : a''^3$.

$3\,a'^3 = a^3$. $\qquad 3\,a''^3 = 2\,a^3$.

$\therefore a' = a\sqrt[3]{\frac{1}{3}}$. $\qquad \therefore a'' = a\sqrt[3]{\frac{2}{3}}$.

Ex. 729. *A lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base that divides the pyramid into two parts which are to each other as 3 : 4?*

$$V : V' = a^3 : a'^3. \qquad \S\ 672$$
$$7 : 3 = a^3 : a'^3.$$
$$7\,a'^3 = 3\,a^3.$$
$$\therefore a' = a\sqrt[3]{\tfrac{3}{7}}.$$

Ex. 730. *The volumes of two similar cones are 54 cubic feet and 432 cubic feet. The height of the first is 6 feet; what is the height of the other?*

$$\sqrt{54} : \sqrt[3]{432} = 6 : H. \qquad \S\ 726$$
$$1 : \sqrt[3]{8} = 6 : H.$$
$$1 : 2 = 6 : H.$$
$$\therefore H = 12 \text{ ft.}$$

Ex. 731. *Two right circular cylinders have their diameters equal to their heights. Their volumes are as 3 : 4. Find the ratio of their heights.*

$$V = \pi R^2 H = 2\pi R^3. \qquad \S\ 700$$
$$V : V' = 2\pi R^3 : 2\pi R'^3.$$
$$1 : \tfrac{4}{3} = R^3 : R'^3.$$
$$R : R' = 1 : \sqrt[3]{\tfrac{4}{3}} = 1 : 0.90856.$$

Ex. 732. *Find the dimensions of a right circular cylinder $\frac{1}{16}$ as large as a similar cylinder whose height is 20 feet, and diameter 10 feet.*

$H : H' = \sqrt[3]{V} : \sqrt[3]{V'}$. § 701 $\quad 2R : 2R' = \sqrt[3]{V} : \sqrt[3]{V'}$. § 701

$20 : H' = 1 : \sqrt[3]{\frac{1}{16}}$. $\quad 10 : 2R' = 1 : \sqrt[3]{\frac{1}{16}}$.

$20 : H' = 1 : 0.97871$. $\quad 10 : 2R' = 1 : 0.97871$.

$\therefore H' = 20 \times 0.97871$ ft. $\quad \therefore 2R' = 10 \times 0.97871$ ft.

$= 19.5742$ ft. $\quad = 9.7871$ ft.

Ex. 733. *The height of a cone of revolution is H, and the radius of its base is R. Find the dimensions of a similar cone three times as large.*

$H : H' = \sqrt[3]{V} : \sqrt[3]{V'}$. § 726 $\quad R : R' = \sqrt[3]{V} : \sqrt[3]{V'}$. § 726

$H : H' = 1 : \sqrt[3]{3}$. $\quad R : R' = 1 : \sqrt[3]{3}$.

$\therefore H' = H\sqrt[3]{3}$. $\quad \therefore R' = R\sqrt[3]{3}$.

Ex. 734. *The height of the frustum of a right cone is $\frac{2}{5}$ the height of the entire cone. Compare the volumes of the frustum and the cone.*

Let V be the volume of the cone,

V' be the volume cut off,

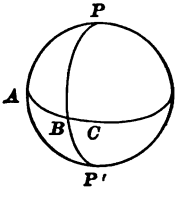
V'' be the volume of the frustum.

$$V : V' = H^3 : H'^3 \qquad \S\ 726$$
$$= 5^3 : 3^3 = 125 : 27.$$
$$V'' : V = 125 - 27 : 125 = 98 : 125.$$

Ex. 735. *The frustum of a pyramid is 8 feet high, and two homologous edges of its bases are 4 feet and 3 feet, respectively. Compare the volume of the frustum and that of the entire pyramid.*

$$V : V' = 4^3 : 3^3 = 64 : 27. \qquad \S\ 672$$
$$V'' : V = 64 - 27 : 64 = 37 : 64.$$

Ex. 736. *Every point in a great circle which bisects a given arc of a great circle at right angles is equidistant from the extremities of the given arc.*



Let AC be any given arc of a great ⊙, and let PBP' be the great ⊙ which bisects the arc AC at right angles.

To prove that every point in the great ⊙ PBP' is equidistant from A and C.

Proof. Draw the chords AB, AC, BC.

Chord AB = chord BC. § 241

⊙ PBP' passes through the poles of arc AC. § 754

But the poles of arc AC are equidistant from A and C, the distances being measured on straight lines.

Hence, the plane of the ⊙ PBP' is ⊥ to the chord AC at its middle point. §§ 496, 517

Therefore, every point in the great ⊙ PBP' is equidistant from the points A and C. Q. E. D.

Ex. 737. *The radius of a sphere is 4 inches. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 inches. Find the area of this circle.*

Let O be the centre of the sphere, P the pole, and A a point in the ⊙.

In the plane $\triangle POA$, $PO = OA = 4$, and $PA = 3$.

Area of $\triangle POA = \sqrt{5\frac{1}{2} \times 1\frac{1}{2} \times 1\frac{1}{2} \times 2\frac{1}{2}}$ Ex. 405

$= 5.56.$

Also area of $\triangle POA = \frac{1}{2} \times 4 \times h = 2h.$ § 403

$\therefore 5.56 = 2h.$

$\therefore r = h = 2.78.$

Area small ⊙ $= \pi r^2$ § 463

$= 3.1416 \times 2.78^2 = 24.28.$

24.28 sq. in. *Ans.*

Ex. 738. *The edge of a regular tetrahedron is a. Find the radii R, R' of the inscribed and circumscribed spheres.*

Let D-ABC be the given regular tetrahedron whose edge is a.

The radius of the ⊙ circumscribed about each face of the tetrahedron is $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{a^3}{4\sqrt{(\frac{3}{2}a)(\frac{1}{2}a)(\frac{1}{2}a)(\frac{1}{2}a)}} = \frac{a}{\sqrt{3}}$. Ex. 315

Let M be the centre of the $\odot$ circumscribed about the $\triangle ACD$, and M' be the centre of the $\odot$ circumscribed about the $\triangle ABC$. Erect the $\perp$s MO and $M'O$ to the faces ACD and ABC, respectively.

In the rt. $\triangle BMC$, $MC = \frac{a}{\sqrt{3}}$, and $BC = a$.

$$\therefore BM = \sqrt{a^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{\frac{2a^2}{3}} = \tfrac{1}{3}a\sqrt{6}. \quad \S\ 372$$

$$\therefore R = OM = \tfrac{1}{4}BM = \tfrac{1}{4} \text{ of } \tfrac{1}{3}a\sqrt{6} = \tfrac{1}{12}a\sqrt{6}.$$

In the rt. $\triangle OMC$, $OM = \frac{1}{12}a\sqrt{6}$, and $MC = \frac{a}{\sqrt{3}}$.

$$\therefore OC = \sqrt{\left(\frac{1}{12}a\sqrt{6}\right)^2 + \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{\frac{9}{24}a^2} = \tfrac{1}{4}a\sqrt{6}. \quad \S\ 371$$

$$\therefore R' = \tfrac{1}{4}a\sqrt{6}.$$

Ex. 739. *Find the diameter of the section of a sphere* 10 *inches in diameter made by a plane* 3 *inches from the centre.*

Let r denote the radius and d the diameter of the $\odot$ made by the intersecting plane. § 743

The plane $\triangle$ formed by drawing the radii of the sphere to the extremities of a diameter of the section is an isosceles $\triangle$ with legs each 5 in. long, and with altitude 3 in.

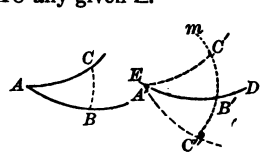
$$\therefore r = \sqrt{5^2 - 3^2} \text{ in.} = \sqrt{16} \text{ in.} = 4 \text{ in.} \quad \S\ 372$$

$$\therefore d = 2r = 2 \times 4 \text{ in.} = 8 \text{ in.}$$

Ex. 740. *At a given point in a given arc of a great circle, to construct a spherical angle equal to a given spherical angle.*

Let A' be any point in the arc ED, and BAC any given $\angle$.

To construct an $\angle$ equal to BAC having A' as its vertex, and $A'D$ as one side.



Construction. With A as pole and a quadrant as radius arc, draw the arc BC.

With A' as a pole, draw the arc of a great $\odot$ $B'm$.

On this arc lay off $B'C'$ equal to BC, and through A' and C' draw the arc of a great $\odot$ $A'C'$. Then $B'A'C'$ is the $\angle$ required.

Proof. The arcs BC and $B'C'$ are the measures of the $\measuredangle$ A and A', respectively. § 779

But $BC = B'C'$. Const.

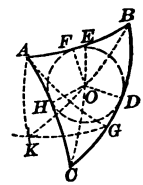
$$\therefore \angle B'A'C' = \angle BAC.$$

Another $\angle B'A'C''$ may be constructed below the line $A'D$. Q. E. F.

Ex. 741. *To inscribe a circle in a given spherical triangle.*

Let ABC be any spherical $\triangle$.

To inscribe a $\odot$ in the spherical $\triangle ABC$.



Construction. With B as pole, and BA as radius arc, describe the arc AG.

From A and G as poles, and with a radius arc greater than $\frac{1}{2}AG$, describe arcs intersecting in K.

Through K and B draw an arc of a great $\odot$.

In like manner, draw CE.

From O, the intersection of BK and CE, let fall the arc of a great $\odot$ $OD \perp$ to the side BC.

With O as pole, and OD as radius arc, draw the $\odot$ DFH.

Then the $\odot$ DFH is the $\odot$ required.

Proof. Draw the arcs of great $\textcircled{S}$ KA and KG.

Lay off BF equal to BD, CH equal to CD, and draw the arcs of great $\textcircled{S}$ OF and OH.

Since the $\triangle$ BAK and BGK are mutually equilateral, Const.

they are therefore mutually equiangular, § 810

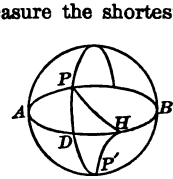
and $\angle OBF = \angle OBD$.

$\therefore$ $\triangle$ OBF and OBD are equal. § 808

Hence, OF is $\perp$ to AB and equal to OD.

In like manner, we may prove that OH is $\perp$ to AC and equal to AD.

It remains to prove that the $\measuredangle$ OD, OF, and OH measure the shortest distances from O to the sides CB, AB, and AC, respectively.



To do this, let PD be the less of the two $\measuredangle$ from P to the arc ADB, and let PH be any other arc of a great $\odot$ from P to ADB.

Then $PD < PH$.

For $HP = HP'$. Ex. 736

$\therefore PD + P'D < PH + HP'$. § 816

$\therefore 2\,PD < 2\,PH$, or $PD < PH$.

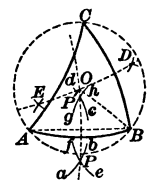
Hence, the $\odot$ described lies within the given $\triangle ABC$, and touches its sides at D, F, and H, respectively. Q. E. F.

Ex. 742. *To circumscribe a circle about a given spherical triangle.*

Let ABC be any spherical $\triangle$.

To circumscribe a $\odot$ about the spherical $\triangle ABC$.

Construction. With A as pole, and radius arc greater than $\frac{1}{2}AB$, draw the arcs ab and cd.



With B as pole, and the same radius arc, draw the arcs cf and gh, intersecting the first arcs in P and P'.

Through P and P' draw the arc of a great $\odot$ $OP'P$.

In like manner, draw the arc of a great $\odot$ DE.

With O, the intersection of PP' and ED, as pole, and OB as radius arc, draw the $\odot$ BCA.

Then the $\odot$ BCA is the $\odot$ required.

Proof. Since P and P' are equidistant from A and B, Const. the distances being measured on arcs of great circles, they are equidistant from A and B, distances being measured on straight lines.

The centre of the sphere is also equidistant from A and B.

Hence, the plane of the arc PP' is perpendicular to the chord AB at its middle point. §§ 496, 517

Therefore, O is equidistant from A and B, distances being measured on straight lines, and, therefore, equidistant, distances being measured on arcs of great ⑤. § 517

In like manner, O is equidistant from B and C.

Hence, the circumference of the $\odot$ described with O as pole, and OB as radius arc, will pass through A, B, and C. Q. E. F.

Ex. 743. *Given a spherical triangle whose sides are* 60°, 80°, *and* 100°. *Find the angles of its polar triangle.*

The $\measuredangle$ of polar $\triangle$ are 120°, 100°, and 80°. § 793

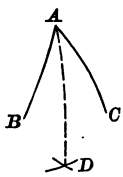
Ex. 744. *Given a spherical triangle whose angles are* 70°, 75°, *and* 95°. *Find the sides of its polar triangle.*

The sides of polar $\triangle$ are 110°, 105°, and 85°. § 793

Ex. 745. *Find the ratio of two homologous sides of two mutually equiangular triangles on spheres whose radii are* 12 *inches and* 20 *inches.*

$12 : 20 = 3 : 5$. § 811, Note

Ex. 746. *To bisect a spherical angle.*



Let BAC be any spherical $\angle$.

To bisect the spherical $\angle$ BAC.

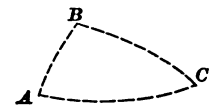
Construction. With A as pole and a quadrant radius arc describe arcs intersecting AB and AC at B and C, respectively.

With B and C as poles and equal radius arcs greater than half the spherical distance BC describe arcs intersecting at D.

Through A and D draw the arc of a great $\odot$.

Then DA bisects the spherical $\angle$ BAC. Q. E. F.

Ex. 747. *To construct a spherical triangle, having given two sides and the included angle.*



Construction. Draw AB, the arc of a great $\odot$, equal to one of the given sides.

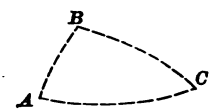
Draw AC, the arc of a great $\odot$, making the $\angle BAC$ equal to the given $\angle$.

Take AC equal to the second of the two given sides.

Draw the arc of a great $\odot$ through B and C.

Then the $\triangle ABC$ is the spherical $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 748. *To construct a spherical triangle, having given two angles and the included side.*

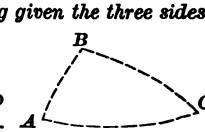


Construction. Draw AC, the arc of a great $\odot$, equal to the given side.

From A and C draw AB and CB, arcs of great $\mathbb{S}$ intersecting at B, making the $\measuredangle CAB$ and ACB equal to the two given $\measuredangle$.

Then the $\triangle ABC$ is the spherical $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 749. *To construct a spherical triangle, having given the three sides.*



Construction. Draw AC, the arc of a great $\odot$, equal to the first of the given sides.

With A and C as poles and radius arcs equal to the second and the third of the given sides respectively, describe arcs intersecting at B.

Through A and B and through C and B pass arcs of great $\mathbb{S}$.

Then the $\triangle ABC$ is the spherical $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 750. *To construct a spherical triangle, having given the three angles.*

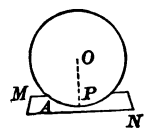
Construction. Now in each of two polar $\measuredangle$ each side of one is the supplement of the $\angle$ opposite in the other. § 793

$\therefore$ construct a spherical $\triangle$ with the three sides equal to the supplements of the given $\measuredangle$. Ex. 749

Then with each vertex of this $\triangle$ as poles and quadrant radii draw arcs of great $\mathbb{S}$ forming a spherical $\triangle$.

This $\triangle$ is the spherical $\triangle$ required, as is obvious from the construction. Q. E. F.

Ex. 751. *To pass a plane tangent to a sphere at a given point on the surface of the sphere.*



Let O be the centre of the given sphere and P a point of the surface.

To pass a plane tangent to the sphere at the point P.

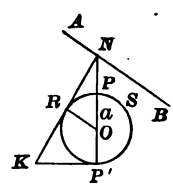
Construction. Draw the radius OP.

Pass the plane $MN \perp$ to OP at P.

Then MN is the plane required. § 765

Q. E. F.

Ex. 752. *To pass a plane tangent to a sphere through a given straight line without the sphere.*



Let AB be any given straight line without the sphere whose centre is O.

To pass a plane through AB tangent to the sphere.

Construction. Through the line AB and the centre O pass a plane.

Let this plane be represented by the plane of the paper.

Its intersection with the sphere is a great ⊙ of the sphere.

Let $P'SPR$ be this great ⊙.

From any point N in AB draw NRK, a tangent to the ⊙ $P'SPR$.

Through N and O draw NOP', and through P' draw $P'K \perp$ to $P'N$.

While AB remains fixed, conceive the figure NKP' to revolve about NP' as an axis.

The semicircle PRP' will generate the given sphere, and the $\triangle NKP'$ will generate the circumscribed circular cone.

Through the line AB pass planes tangent to this cone.

Then these two planes are the required tangent planes to the sphere.

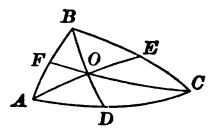
Proof. Every element of the cone has one, and only one, point in common with the surface of the sphere.

Hence, any plane tangent to the cone and embracing AB and only one of these elements has one, and only one, point in common with the surface of the sphere, and is, therefore, a tangent plane to the sphere. § 741

Q. E. F.

Ex. 753. *The three medians of a spherical triangle meet in a point.*

Let ABC be any spherical $\triangle$, and let AE, BD, and CF be the medians of the $\triangle ABC$.



To prove that AE, BD, and CF intersect in the same point.

Proof. The chords AE, BD, and CF, being the medians of the plane $\triangle ABC$, intersect in a point. Ex. 27

$\therefore$ the arcs AE, BD, and CF intersect in a point. Q. E. D.

Ex. 754. *To construct a spherical surface with a given radius that passes through three given points.*

Let R be the given radius, and A, B, and C any three given points.

To construct a spherical surface with radius R, which shall contain A, B, and C.

Analysis. The centre of the sphere must be at the distance R from each of the points A, B, and C; hence it is in the surface of the spheres having A, B, and C as centres, and R as radius.

Construction. Construct a sphere with A as centre and R as radius, also spheres with B and C as centres and R as radius.

The spheres having as centres the points common to the surfaces of the three spheres, and the given radius R, will embrace in its surface the three given points.

Proof. The distances of the points A, B, and C from either of these common points are equal to the given radius R; hence these points are in the surfaces of spheres having these points as centres and R as radius.

Q. E. F.

Discussion. If one of the three spheres cuts the intersection of the surfaces of the other two, there will be two solutions.

If one simply touches the intersection, or the point of contact of the other two, there will be one solution.

If the surfaces of the three spheres have no point in common, there will be no solution.

Ex. 755. *To construct a spherical surface with a given radius that passes through two given points and is tangent to a given plane.*

Let R be the given radius, MN the given plane, and A and B the two given points.

To construct a spherical surface with radius R through the points A and B tangent to the plane MN.

Analysis. The centre of the sphere must be at the distance R from each of the points A and B.

Hence, it is in the surfaces of the spheres having A and B as centres and R as radius.

The centre must also be at the distance R from the plane MN.

Hence, it is in one of the planes $\parallel$ to MN at the distance R.

Construction. Construct a sphere with A as centre and R as radius, also a sphere with B as centre and R as radius.

Construct also the planes $\parallel$ to the plane MN at the distance R.

The spheres having as centres the points common to the surfaces of the two spheres and either of the two planes, and the given radius R will pass through the two points and be tangent to the given plane.

Proof. The surface of each of these spheres passes through the points A and B, since by construction A and B are at the distance R from its centre.

The plane MN is tangent to each sphere, since by construction R is the $\perp$ distance of its centre from the plane MN. Q. E. F.

Discussion. If the intersection of the spheres A and B lies in one of the planes $\parallel$ to MN, the problem is indeterminate.

If one of the planes $\parallel$ to MN cuts the intersection of the two spherical surfaces, there will be two solutions.

If one of these planes touches this intersection, there will be one solution.

If the two spheres have no point in common with either plane, there will be no solution.

If the points are on opposite sides of the plane, or if both are in the plane, there will be no solution.

Ex. 756. *To construct a spherical surface with a given radius that passes through two given points and is tangent to a given sphere.*

Let R be the given radius, A and B any two given points, and O the centre of a sphere whose radius is R'.

To construct a spherical surface with radius R through the points A and B tangent to the sphere O.

Analysis. The centre of the sphere must be at the distance R from each of the points A and B.

Hence, it is in the surfaces of the spheres having A and B as centres and R as radius.

That the constructed sphere may be tangent to the given sphere the distance between their centres must equal the sum or difference of their radii.

Construction. Construct the spheres with A and B as centres and R as a common radius.

Construct also the spheres with O as a common centre and with $R + R'$ and $R \frown R'$ as radii.

The spheres having as centres the points common to the surfaces of the first two spheres and that of either of the second two, and the given radius R, will pass through the two points and be tangent to the given sphere.

Proof. The surface of each of these spheres will pass through the points A and B, since by construction A and B are at the distance R from its centre.

The sphere O and each of the spheres drawn will be tangent, since by construction the distance between their centres is equal to the sum or the difference of their radii. Q. E. F.

Discussion. There will be four, three, two, one, or no solution, according as the spherical surfaces A and B have four, three, two, one, or no points in common with the constructed spherical surfaces having O as their centre.

Ex. 757. *All arcs of great circles drawn through a pole of a given great circle are perpendicular to the circumference of the great circle.*

Let P be the pole of the great $\odot ABC$, and let PA be an arc of a great $\odot$ drawn through P intersecting $\odot ABC$ at A.

To prove that PA is $\perp$ to ABC.

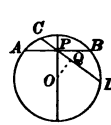
Proof. Now the measure of the spherical $\angle PAB$ is the arc of a great $\odot$ described with A as pole and included between its sides. § 779

$\therefore$ the $\angle PAB$ is measured by a quadrant (the distance from P to the $\odot ABC$).

$\therefore \angle PAB$ is a rt. spherical $\angle$. Q. E. D.

Ex. 758. *The smallest circle whose plane passes through a given point within a sphere is the one whose plane is perpendicular to the radius through the given point.*

Let P be a point within the sphere whose centre is O.



To prove that the smallest $\odot$ through P is $\perp$ to the radius through P.

Proof. Pass a plane through P and the centre of the sphere.

The chord $AB \perp$ to OP is the shortest chord that can be drawn through P in this great $\odot$. Ex. 109

Hence, AB is a diameter of the smallest $\odot$ through P.

$\therefore$ the smallest $\odot$ through P is $\perp$ to the radius through P. Q. E. D.

Ex. 759. *Given the radius of a sphere* 10 *inches. Find the area of a lune whose angle is* 30°.

$$L = \frac{\pi R^2 A}{90} = \frac{3.1416 \times 10^2 \times 30}{90} = 104.72. \qquad \S\ 831$$

104.72 sq. in. *Ans.*

Ex. 760. *Given the diameter of a sphere* 16 *inches. Find the area of a lune whose angle is* 75°.

$$L = \frac{\pi R^2 A}{90} = \frac{3.1416 \times 8^2 \times 75}{90} = 167.552. \qquad \S\ 831$$

167.55 sq. in. *Ans.*

Ex. 761. *What part of the surface of a sphere is a triangle whose angles are* 120°, 100°, *and* 95°? *What is its area in square inches, if the radius of the sphere is* 6 *inches?*

$$E = 120° + 100° + 95° - 180° = 135°.$$

$$\frac{\Delta}{S} = \frac{135}{720} = \frac{3}{16}. \qquad \S\ 835$$

$$\Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 6^2 \times 135}{180} = 84.8232. \qquad \S\ 836$$

84.82 sq. in. *Ans.*

Ex. 762. *Find the area of a spherical triangle whose angles are* 100°, 120°, 140°, *if the diameter of the sphere is* 16 *inches.*

$$E = 100° + 120° + 140° - 180° = 180°.$$

$$\Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 8^2 \times 180}{180} = 201.0624. \qquad \S\ 836$$

201.06 sq. in. *Ans.*

Ex. 763. *If the radii of two spheres are* 6 *inches and* 4 *inches respectively, and the distance between their centres is* 5 *inches, what is the area of the circle of intersection of these spheres?*

Let S denote the area of the Δ formed by the line of centres and the radii to any point of the circumference of the section.

$$S = \sqrt{\tfrac{15}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times \tfrac{7}{2}} = \tfrac{15}{4}\sqrt{7}. \qquad \text{Ex. } 405$$

$$S = \tfrac{5}{2} R. \qquad \S\ 403$$

$$\therefore \tfrac{5}{2} R = \tfrac{15}{4}\sqrt{7}.$$

$$\therefore R = \tfrac{3}{2}\sqrt{7}.$$

$$\text{Area of circle} = \pi R^2 = 3.1416 \times (\tfrac{3}{2}\sqrt{7})^2 = 49.4802. \qquad \S\ 463$$

49.48 sq. in. *Ans.*

Ex. 764. *Find the radius of the circle determined in a sphere of* 5 *inches diameter by a plane* 1 *inch from the centre.*

$$R = \sqrt{(2\tfrac{1}{2})^2 - 1^2} \text{ in.} = \tfrac{1}{2}\sqrt{25 - 4} \text{ in.} = \tfrac{1}{2}\sqrt{21} \text{ in.} \qquad \S\ 372$$

Ex. 765. *If the radii of two concentric spheres are R and R', and if a plane is drawn tangent to the interior sphere, what is the area of the section made in the other sphere?*

Let R be the radius of the larger sphere.

$$r = \sqrt{R^2 - R'^2}. \qquad \S\ 372$$

$$\text{Area} = \pi r^2 = \pi (R^2 - R'^2). \qquad \S\ 463$$

Ex. 766. *Two points A and B are* 8 *inches apart. Find the locus in space of a point* 5 *inches from A and* 7 *inches from B.*

Let R denote the $\perp$ from any point C of the locus to AB.

Then $$R = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{Ex. 312}$$

$$= \tfrac{2}{8}\sqrt{10 \times 3 \times 5 \times 2} = \tfrac{1}{4} \text{ of } 10\sqrt{3} = \tfrac{5}{2}\sqrt{3}.$$

Hence, the locus is the circumference described with R as radius and the foot of R as centre.

Ex. 767. *The radii of two parallel sections of the same sphere are a and b, respectively, and the distance between these sections is d. Find the radius of the sphere.*

Let OA and OB be the $\perp$s from the centres to the planes, respectively.

$$OA = \sqrt{R^2 - a^2}; \quad OB = \sqrt{R^2 - b^2}.$$

$$OA - OB = d = \sqrt{R^2 - a^2} - \sqrt{R^2 - b^2}.$$

$$\therefore d + \sqrt{R^2 - b^2} = \sqrt{R^2 - a^2}.$$

$$d^2 + 2d\sqrt{R^2 - b^2} + R^2 - b^2 = R^2 - a^2.$$

$$2d\sqrt{R^2 - b^2} = b^2 - a^2 - d^2.$$

$$4d^2R^2 - 4b^2d^2 = a^4 + b^4 + d^4 - 2a^2b^2 - 2b^2d^2 + 2a^2d^2.$$

$$4d^2R^2 = a^4 + b^4 + d^4 + 2b^2d^2 + 2a^2d^2 - 2a^2b^2.$$

$$R = \frac{1}{2d}\sqrt{a^4 + b^4 + d^4 + 2b^2d^2 + 2a^2d^2 - 2a^2b^2}.$$

Ex. 768. *Find the surface of a sphere if the diameter is (i)* 10 *inches; (ii)* 1 *foot* 9 *inches; (iii)* 2 *feet* 4 *inches; (iv)* 7 *feet; (v)* 10.5 *feet.*

(*i*) $$S = 4\pi R^2 = 4 \times 3.1416 \times 5^2 = 314.16. \qquad \S\ 824$$

314.16 sq. in. *Ans.*

(*ii*) 1 ft. 9 in. $= 1\frac{3}{4}$ ft.

$S = 4\pi R^2 = 4 \times 3.1416 \times (\frac{7}{8})^2 = 9.6212.$ § 824

9.62 sq. ft. *Ans.*

(*iii*) 2 ft. 4 in. $= 2\frac{1}{3}$ ft.

$S = 4\pi R^2 = 4 \times 3.1416 \times (\frac{7}{6})^2 = 17.1043.$ § 824

17.10 sq. ft. *Ans.*

(*iv*) $S = 4\pi R^2 = 4 \times 3.1416 \times (\frac{7}{2})^2 = 153.9384.$ § 824

153.94 sq. ft. *Ans.*

(*v*) $S = 4\pi R^2 = 4 \times 3.1416 \times (5.25)^2 = 346.3614.$ § 824

346.36 sq. ft. *Ans.*

Ex. 769. *Find the diameter of a sphere if the surface is* (*i*) 616 *square inches;* (*ii*) $38\frac{1}{2}$ *square feet;* (*iii*) 9856 *square feet.*

(*i*) $S = 4\pi R^2 = \pi D^2.$ § 824

$616 = 3.1416 \times D^2.$

$D^2 = 616 \times 0.31831$ § 481

$= 196.07896.$

$\therefore D = \sqrt{196.07896} = 14.$

$D = 14$ in.

(*ii*) $S = \pi D^2.$

$38\frac{1}{2} = 3.1416 \times D^2.$

$D^2 = 38\frac{1}{2} \times 0.31831$

$= 12.254935.$

$\therefore D = \sqrt{12.254935} = 3.5.$

$D = 3.5$ ft.

(*iii*) $S = \pi D^2.$

$9856 = 3.1416 \times D^2.$

$D^2 = 9856 \times 0.31831$

$= 3137.26336.$

$\therefore D = \sqrt{3137.26336} = 56.$

$D = 56$ ft.

Ex. 770. *The circumference of a dome in the shape of a hemisphere is 66 feet. How many square feet of lead are required to cover it?*

$S = 2\pi R^2 = 66\,R.$ § 824

$R = \dfrac{66}{2\pi} = 33 \times 0.31831.$ § 458

$S = 66 \times 33 \times 0.31831 = 693.27918.$

693.28 sq. ft. *Ans.*

Ex. 771. *If the ball on the top of St. Paul's Cathedral in London is 6 feet in diameter, what would it cost to gild it at 7 cents per square inch?*

$S = \pi D^2 = 3.1416 \times (6)^2$ sq. ft. § 824

$= 113.0976$ sq. ft.

$= 16{,}286.0544$ sq. in.

Cost $= 16{,}286.0544 \times \$0.07 = \$1140.02.$

Ex. 772. *What is the numerical value of the radius of a sphere if its surface has the same numerical value as the circumference of a great circle?*

$$S = 4\pi R^2. \qquad \S\ 824$$
$$C = 2\pi R. \qquad \S\ 458$$
$$\therefore 4\pi R^2 = 2\pi R.$$
$$2R = 1.$$
$$R = \tfrac{1}{2}.$$

Ex. 773. *Find the surface of a lune, if its angle is* 30°, *and the total surface of the sphere is* 4 *square feet.*

$$L : S = 30 : 360. \qquad \S\ 829$$
$$L : 4 = 1 : 12.$$
$$\therefore L = \tfrac{4}{12} \text{ sq. ft.} = \tfrac{1}{3} \text{ sq. ft.}$$

Ex. 774. *What fractional part of the whole surface of a sphere is a spherical triangle whose angles are* 43° 27′, 81° 57′, *and* 114° 36′?

$$E = 43°\,27' + 81°\,57' + 114°\,36' - 180° = 60°.$$
$$\frac{\Delta}{S} = \frac{E}{720} = \frac{60}{720} = \frac{1}{12}. \qquad \S\ 835$$

Ex. 775. *The angles of a spherical triangle are* 60°, 70°, *and* 80°. *The radius of the sphere is* 14 *feet. Find the area of the triangle.*

$$E = 60° + 70° + 80° - 180° = 30°.$$
$$\text{Area } \Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 14^2 \times 30}{180} = 102.6256. \qquad \S\ 836$$
102.63 sq. ft. *Ans.*

Ex. 776. *The sides of a spherical triangle are* 80°, 74°, *and* 128°. *The radius of the sphere is* 14 *feet. Find the area of the polar triangle in square feet.*

The ∡s of the polar Δ are 100°, 106°, 52°. § 793

$$\therefore E = 100° + 106° + 52° - 180° = 78°.$$
$$\text{Area polar } \Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 14^2 \times 78}{180} = 266.82656. \qquad \S\ 836$$
266.83 sq. ft. *Ans.*

Ex. 777. *Find the area of a spherical polygon on a sphere whose radius is* $10\frac{1}{2}$ *feet, if its angles are* 100°, 120°, 140°, *and* 160°.

$$E = T - (n - 2)\,180°.$$
$$T = 100° + 120° + 140° + 160° = 520°.$$

$$E = 520° - (2 \times 180°)$$
$$= 520° - 360° = 160°.$$
$$\frac{\text{Area}}{S} = \frac{160}{720} = \frac{2}{9}. \qquad \S\ 837$$
$$S = 4\pi R^2 = 4 \times 3.1416 \times 10.5 \times 10.5. \qquad \S\ 824$$
$$\therefore \text{Area} = \frac{2 \times 4 \times 3.1416 \times 10.5 \times 10.5}{9} = 307.8768.$$

307.88 sq. ft. *Ans.*

Ex. 778. *The planes of the faces of a quadrangular spherical pyramid make with each other angles of* 80°, 100°, 120°, *and* 150°; *and the length of a lateral edge of the pyramid is* 42 *feet. Find the area of its base in square feet.*

$$E = T - (n - 2)\,180°.$$
$$T = 80° + 100° + 120° + 150° = 450°.$$
$$E = 450° - (2 \times 180°)$$
$$= 450° - 360° = 90°.$$
$$\frac{\text{Area}}{S} = \frac{90}{720} = \frac{1}{8}. \qquad \S\ 837$$
$$S = 4\pi R^2 = 4 \times 3.1416 \times 42^2. \qquad \S\ 824$$
$$\therefore \text{Area} = \frac{4 \times 3.1416 \times 42^2}{8} = 2770.8912.$$

2770.89 sq. ft. *Ans.*

Ex. 779. *The planes of the faces of a triangular spherical pyramid make with each other angles of* 60°, 80°, *and* 100°, *and the area of the base of the pyramid is* 4π *square feet. Find the radius of the sphere.*

$$E = 60° + 80° + 100° - 180° = 60°.$$
$$\text{Area } \triangle = \frac{\pi R^2 E}{180} = \frac{\pi R^2 \times 60}{180} = \tfrac{1}{3}\pi R^2. \qquad \S\ 836$$
$$\therefore 4\pi = \tfrac{1}{3}\pi R^2.$$
$$\therefore R^2 = 12.$$
$$\therefore R = \sqrt{12} = 3.46.$$

3.46 ft. *Ans.*

Ex. 780. *The diameter of a sphere is* 21 *feet. Find the curved surface of a segment whose height is* 5 *feet.*

$$\text{Area} = 2\pi RH = 2 \times 3.1416 \times 10.5 \times 5 = 329.8680. \qquad \S\ 826$$

329.87 sq. ft. *Ans.*

Ex. 781. *In a sphere whose radius is R, find the height of a zone whose area is equal to that of a great circle.*

$$Z = 2\pi RH. \qquad \S\ 826$$
$$\text{Great circle} = \pi R^2. \qquad \S\ 463$$
$$\therefore 2\pi RH = \pi R^2.$$
$$2H = R.$$
$$H = \tfrac{1}{2}R.$$

Ex. 782. *What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?*

$$\text{Area} = \pi \times (\text{chord})^2 = \pi \times (\sqrt{h^2 + r^2})^2 = \pi(h^2 + r^2). \qquad \S\ 828$$

If the height were twice as great, the area would be $2\pi(h^2 + r^2)$.

Ex. 783. *The altitude of the torrid zone is 3200 miles. Find its area, assuming the earth to be a sphere with a radius of 4000 miles.*

$$Z = 2\pi RH = 2 \times 3.1416 \times 4000 \times 3200 = 80{,}424{,}960. \qquad \S\ 826$$

80,424,960 sq. mi. *Ans.*

Ex. 784. *A plane divides the surface of a sphere of radius R into two zones, such that the surface of the greater is the mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere.*

The altitudes of the two zones are $2R - H$ and H.

Then
$$\frac{2\pi R(2R-H)}{2\pi RH} = \frac{2\pi RH}{4\pi R^2}. \qquad \S\S\ 826, 824$$
$$\frac{2R-H}{H} = \frac{H}{2R}.$$
$$4R^2 - 2RH = H^2.$$
$$H^2 + 2RH + R^2 = 5R^2.$$
$$H + R = R\sqrt{5}.$$
$$H = R\sqrt{5} - R.$$
$$\text{Distance from centre} = R\sqrt{5} - R - R = R(\sqrt{5} - 2).$$

Ex. 785. *If a sphere of radius R is cut by two parallel planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre.*

Let a denote the radius of each section.

Then
$$a = \sqrt{R^2 - \tfrac{1}{4}H^2}. \qquad \S\ 372$$

$$Z = 2\pi RH. \qquad \S\ 826$$

$$2B = 2\pi a^2 = 2\pi(R^2 - \tfrac{1}{4}H^2). \qquad \S\ 463$$

$$\therefore 2\pi RH = 2\pi(R^2 - \tfrac{1}{4}H^2).$$

$$4\pi RH = 4\pi R^2 - \pi H^2.$$

$$H^2 + 4RH = 4R^2.$$

$$H^2 + 4RH + 4R^2 = 8R^2.$$

$$H + 2R = 2R\sqrt{2}.$$

$$H = 2R\sqrt{2} - 2R.$$

$$\tfrac{1}{2}H = R\sqrt{2} - R = R(\sqrt{2} - 1).$$

Ex. 786. *Find the area of the zone generated by an arc of* 30°, *of which the radius is* r, *and which turns around a diameter passing through one of its extremities.*

Let AC denote the radius and AB the distance from the centre to the section.

Then $$AB = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} = \sqrt{\tfrac{3}{4}r^2} = \frac{r}{2}\sqrt{3}. \qquad \S\ 372$$

$$BC = r - \frac{r}{2}\sqrt{3}.$$

That is, $$H = \frac{r}{2}(2 - \sqrt{3}).$$

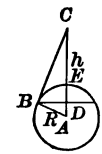
$$Z = 2\pi RH \qquad \S\ 826$$

$$= 2\pi r \times \frac{r}{2}(2 - \sqrt{3}) = \pi r^2(2 - \sqrt{3}).$$

Ex. 787. *Find the area of the zone of a sphere of radius* R, *illuminated by a lamp placed at the distance* h *from the sphere.*

In the similar $\triangle ADB$ and ABC,

$$\frac{AD}{AB} = \frac{AB}{AC}. \qquad \S\ 351$$



That is, $$\frac{AD}{R} = \frac{R}{R+h}.$$

$$AD = \frac{R^2}{R+h},$$

$$DE = R - \frac{R^2}{R+h} = \frac{Rh}{R+h}.$$

$$Z = 2\pi R \times \frac{Rh}{R+h} = \frac{2\pi R^2 h}{R+h}. \qquad \S\ 826$$

Ex. 788. *How much of the earth's surface would a man see if he were raised to the height of the radius above it?*

$$Z = \frac{2\pi R^2 h}{R+h}. \qquad \text{Ex. 787}$$

But $h = R.$

$$\therefore Z = \frac{2\pi R^3}{2R} = \pi R^2.$$

$$S = 4\pi R^2. \qquad \S\ 824$$

$$\frac{Z}{S} = \frac{\pi R^2}{4\pi R^2} = \frac{1}{4}.$$

Ex. 789. *To what height must a man be raised above the earth in order that he may see one sixth of its surface?*

$$S = 4\pi R^2. \qquad \S\ 824$$

$$\therefore Z = \frac{S}{6} = \tfrac{2}{3}\pi R^2.$$

But $$Z = \frac{2\pi R^2 h}{R+h}. \qquad \text{Ex. 787}$$

$$\therefore \frac{2\pi R^2 h}{R+h} = \frac{2\pi R^2}{3}.$$

That is, $$\frac{h}{R+h} = \frac{1}{3}.$$

$$\therefore 3h = R + h.$$

$$\therefore h = \tfrac{1}{2} R.$$

Ex. 790. *The square on the diameter of a sphere and the square on an edge of the inscribed cube are as* 3 : 1.

Let a denote an edge of the inscribed cube.

The diameter of the sphere $= 2R =$ the diagonal of the cube.

But the diagonal of the cube $= a\sqrt{3}.$ Ex. 653

$$\therefore 2R = a\sqrt{3}.$$

$$\therefore (2R)^2 : a^2 = (a\sqrt{3})^2 : a^2 = 3a^2 : a^2 = 3 : 1.$$

Ex. 791. *Find the volume of a sphere if the diameter is* (*i*) 13 *inches;* (*ii*) 3 *feet* 6 *inches;* (*iii*) 10 *feet* 6 *inches;* (*iv*) 14.7 *feet.*

(*i*) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times 13^3 = 1150.3492.$ § 845

1150.35 cu. in. *Ans.*

(*ii*) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times (\frac{7}{2})^3 = 22.44935.$ § 845

22.45 cu. ft. *Ans.*

(*iii*) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times (8\frac{1}{2})^3 = 606.13245.$ § 845

606.13 cu. ft. *Ans.*

(*iv*) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times (14.7)^3 = 1663.2274428.$ § 845

1663.23 cu. ft. *Ans.*

Ex. 792. *Find the diameter of a sphere if the volume is* (*i*) 75 *cubic feet* 1377 *cubic inches*; (*ii*) 179 *cubic feet* 1152 *cubic inches*; (*iii*) 1047.816 *cubic feet*; (*iv*) 38.808 *cubic yards.*

(*i*) 75 cu. ft. 1377 cu. in. = 130,977 cu. in.

$V = \frac{1}{6}\pi D^3.$ § 845

$130{,}977 = \frac{1}{6} \times 3.1416\, D^3.$

$D^3 = 6 \times 0.31831 \times 130977 = 250{,}147.73322.$

$D = 63$ in.

(*ii*) 179 cu. ft. 1152 cu. in. = 310464 cu. in.

$V = \frac{1}{6}\pi D^3.$ § 845

$310{,}464 = \frac{1}{6} \times 3.1416\, D^3.$

$D^3 = 6 \times 0.31831 \times 310464 = 592{,}942.77504.$

$D = 84$ in.

(*iii*) $V = \frac{1}{6}\pi D^3.$ § 845

$1047.816 = \frac{1}{6} \times 3.1416\, D^3.$

$D^3 = 6 \times 0.31831 \times 1047.816 = 2001.18186576.$

$D = 12.6$ ft.

(*iv*) $V = \frac{1}{6}\pi D^3.$ § 845

$38.808 = \frac{1}{6} \times 3.1416\, D^3.$

$D^3 = 6 \times 0.31831 \times 38.808 = 74.11784688.$

$D = 4.2$ yd.

Ex. 793. *Find the volume of a sphere whose circumference is* 45 *feet.*

$$D = \frac{45}{\pi}.$$

$$V = \tfrac{1}{6}\pi D^3 = \frac{1}{6} \times \pi \times \frac{45^3}{\pi^3} = \frac{1}{6} \times 0.31831^2 \times 45^3 = 1538.8.$$ § 845

1538.8 cu. ft. *Ans.*

Ex. 794. *Find the volume V of a sphere in terms of the circumference C of a great circle.*

$V = \frac{1}{6}\pi D^3.$ § 845

$C = \pi D.$ § 458

$$\therefore D = \frac{C}{\pi}.$$

$$\therefore V = \frac{\pi}{6} \times \frac{C^3}{\pi^3} = \frac{C^3}{6\,\pi^2}.$$

Ex. 795. *Find the radius R of a sphere, having given the volume V.*

$$V = \tfrac{4}{3}\pi R^3. \qquad \S\ 845$$

$$\therefore R^3 = \frac{3V}{4\pi}.$$

$$\therefore R = \sqrt[3]{\frac{3V}{4\pi}}.$$

Ex. 796. *Find the radius R of a sphere, if its circumference and its volume have the same numerical value.*

$$\tfrac{4}{3}\pi R^3 = 2\pi R.$$

$$\tfrac{4}{3}R^2 = 1.$$

$$2R^2 = 3.$$

$$R^2 = 1.5.$$

$$\therefore R = \sqrt{1.5} = \tfrac{1}{2}\sqrt{6} = 1.2247.$$

Ex. 797. *The volume of a sphere is to the volume of the circumscribed cube as π is to 6.*

Let a denote the edge of the circumscribed cube.

Then $a = 2R$.

Volume of sphere $= \tfrac{4}{3}\pi R^3$. $\qquad \S\ 845$

$$\therefore \tfrac{4}{3}\pi R^3 : a^3 = \tfrac{4}{3}\pi R^3 : (2R)^3 = \tfrac{4}{3}\pi R^3 : 8R^3 = \pi : 6.$$

Ex. 798. *An iron ball 4 inches in diameter weighs 9 pounds. Find the weight of an iron shell 2 inches thick, whose external diameter is 20 inches.*

$$V : V'' = 20^3 : 4^3 = 5^3 : 1^3 = 125 : 1. \qquad \S\ 846$$

$\therefore V$ weighs 125×9 lb. $= 1125$ lb.

$$V' : V'' = 16^3 : 4^3 = 4^3 : 1 = 64 : 1. \qquad \S\ 846$$

$\therefore V'$ weighs 64×9 lb. $= 576$ lb.

$V - V' = 1125$ lb. $- 576$ lb. $= 549$ lb.

Ex. 799. *The radius of a sphere is 7 feet. What is the volume of a wedge whose angle is 36°?*

$$\frac{W}{V} = \frac{36°}{360°} = \frac{1}{10}.$$

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3} \times 3.1416 \times 7^3 = 1436.7584. \qquad \S\ 845$$

$\therefore W = \frac{1}{10}$ of 1436.76 cu. ft. $= 143.68$ cu. ft.

Ex. 800. *What is the angle of a spherical wedge, if its volume is one cubic foot, and the volume of the entire sphere is 6 cubic feet?*

$$\frac{W}{V} = \frac{1}{6} = \frac{\text{angle of wedge}}{360°}.$$

$\therefore$ angle of wedge $= \frac{1}{6}$ of $360° = 60°$.

Ex. 801. *Find the volume of a spherical sector, if the area of the zone of its base is* 3 *square feet, and the radius of the sphere is* 1 *foot.*

$$V = \tfrac{1}{3}RZ = \tfrac{1}{3} \times (1 \times 3) \text{ cu. ft.} = 1 \text{ cu. ft.} \qquad \S\ 848$$

Ex. 802. *The radius of the base of a segment of a sphere is* 16 *inches, and the radius of the sphere is* 20 *inches. Find the volume of the segment.*

The $\perp$ from the centre of the sphere to the base

$$= \sqrt{20^2 - 16^2} = \sqrt{144} = 12. \qquad \S\ 372$$

$$\therefore a = 20 - 12 = 8.$$

$$V = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3 = \tfrac{1}{2} \times 3.1416 \times 16^2 \times 8 + \tfrac{1}{6} \times 3.1416 \times 8^3 \qquad \S\ 849$$

$$= 3216.9984 + 268.0832$$

$$= 3485.0816. \qquad 3485.08 \text{ cu. in. } \textit{Ans.}$$

Ex. 803. *The inside of a wash-basin is in the shape of a segment of a sphere; the distance across the top is* 16 *inches, and its greatest depth is* 6 *inches. Find how many pints of water it will hold, reckoning* $7\frac{1}{2}$ *gallons to the cubic foot.*

$$V = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3 \qquad \S\ 849$$

$$= \tfrac{1}{2} \times 3.1416 \times 8^2 \times 6 + \tfrac{1}{6} \times 3.1416 \times 6^3$$

$$= 3.1416 \times (192 + 36)$$

$$= 3.1416 \times 228$$

$$= 716.2848.$$

$$716.2848 \text{ cu. in.} = \frac{716.2848}{1728} \text{ cu. ft.} = \frac{716.2848}{1728} \times 7\tfrac{1}{2} \times 8 \text{ pt.}$$

$$= 24.87 \text{ pt.}$$

Ex. 804. *What is the height of a zone, if its area is* S, *and the volume of the sphere to which it belongs is* V?

$$V = \tfrac{4}{3}\pi R^3. \qquad \S\ 845$$

$$S = 2\pi RH. \qquad \S\ 826$$

$$\therefore H = \frac{S}{2\pi R},$$

$$\text{and } R = \sqrt[3]{\frac{3V}{4\pi}}.$$

$$2\pi R = 2\pi \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{6\pi^2 V}.$$

$$\therefore H = \frac{S}{\sqrt[3]{6\pi^2 V}}.$$

Ex. 805. *The radii of the bases of a spherical segment are 6 feet and 8 feet, and its height is 3 feet. Find its volume.*

$$V = \frac{H}{2}(\pi r^2 + \pi r'^2) + \frac{\pi H^3}{6} = \tfrac{3}{2}(\pi 6^2 + \pi 8^2) + \frac{\pi 3^3}{6} \qquad \text{§ 849}$$

$$= 3.1416 \times (150 + 4.5)$$

$$= 485.3772. \qquad 485.38 \text{ cu. ft. } \textit{Ans.}$$

Ex. 806. *Find the volume of a triangular spherical pyramid if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 feet.*

$$E = 3 \times 100° - 180° = 120°.$$

$$\text{Area } \Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 7^2 \times 120}{180} = 102.6256. \qquad \text{§ 836}$$

$$V = \tfrac{1}{3} R \times B = \tfrac{1}{3} \times 7 \times 102.6256 = 239.4597. \qquad \text{§ 847}$$

239.46 cu. ft. *Ans.*

Ex. 807. *The circumference of a sphere is 28π feet. Find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the dihedral angles of which are 80°, 105°, and 140°.*

$$C = 2\pi R = 28\pi. \qquad \text{§ 458}$$

$$\therefore R = 14.$$

$$E = 80° + 105° + 140° - 180° = 145°.$$

$$\text{Area } \Delta = \frac{\pi R^2 E}{180} = \frac{3.1416 \times 14^2 \times 145}{180} = 496.0237. \qquad \text{§ 836}$$

$$V = \tfrac{1}{3} R \times B = \tfrac{14}{3} \times 496.0237 = 2314.7773. \qquad \text{§ 847}$$

2314.78 cu. ft. *Ans.*

Ex. 808. *The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is $3\frac{1}{2}$ feet. Find the volume of the pyramid.*

$$E = 80° + 100° + 120° + 150° - 360° = 90°.$$

$$\frac{\text{Area}}{S} = \frac{90}{720} = \frac{1}{8}. \qquad \text{§ 837}$$

$$S = 4\pi R^2 = 4 \times 3.1416 \times (\tfrac{7}{2})^2. \qquad \text{§ 824}$$

$$\therefore \text{Area} = \tfrac{1}{8} \times 4 \times 3.1416 \times (\tfrac{7}{2})^2.$$

$$V = \tfrac{1}{3} R \times B = \tfrac{1}{3} \times \tfrac{7}{2} \times \tfrac{1}{8} \times 4 \times 3.1416 \times (\tfrac{7}{2})^2$$

$$= 22.44935. \qquad \text{§ 847}$$

22.45 cu. ft. *Ans.*

Ex. 809. *Having given the volume V, and the height h, of a spherical segment of one base, find the radius R of the sphere.*

$$V = \pi h^2\left(R - \frac{h}{3}\right) = \pi h^2 R - \tfrac{1}{3}\pi h^3. \qquad \text{§ 849}$$

$$\therefore R = \frac{V + \tfrac{1}{3}\pi h^3}{\pi h^2} = \frac{V}{\pi h^2} + \frac{h}{3}.$$

Ex. 810. *Find the weight of a sphere of radius R, which floats in a liquid of specific gravity s, with one fourth of its surface above the surface of the liquid.* (*The weight of a floating body is equal to the weight of the liquid displaced.*)

$$Z = 2\pi RH. \qquad \text{§ 826}$$

$$\tfrac{1}{4}S = \pi R^2 = 2\pi RH. \qquad \text{§ 824}$$

$$\therefore H = \tfrac{1}{2}R.$$

The volume out of the liquid $= \pi H^2\left(R - \dfrac{H}{3}\right)$ § 849

$$= \pi\frac{R^2}{4}\left(R - \frac{R}{6}\right)$$

$$= \frac{\pi R^2}{4} \times \frac{5R}{6} = \frac{5\pi R^3}{24}.$$

Volume of sphere $= \tfrac{4}{3}\pi R^3$.

Therefore the volume of the liquid displaced

$$= \tfrac{4}{3}\pi R^3 - \tfrac{5}{24}\pi R^3 = \tfrac{9}{8}\pi R^3.$$

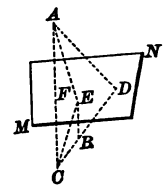
This weighs $\tfrac{9}{8}\pi R^3 s$.

MISCELLANEOUS EXERCISES.

Ex. 811. *Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.*

Let A and B be any two points on opposite sides of the plane MN.

To find the point in the plane MN the difference of whose distances from A and B shall be a maximum.



Construction. From A draw AF perpendicular to the plane MN, and prolong it to C, making FC equal to AF.

Through C and B draw a line meeting the plane in some point D.

Then D is the required point.

Proof. Let E be any other point of the plane MN.

Draw EA, EC, EB, and AD.

Then $DC = DA$,

and $EC = EA$. § 517

$\therefore DA - DB = DC - DB = BC$.

Also $EA - EB = EC - EB < BC$. § 138

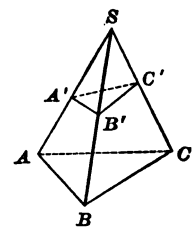
Hence, $DA - DB > EA - EB$. Q. E. F.

Discussion. If the points A and B are equally distant from the plane, the difference of distances is the constant zero, and there is no maximum difference of distances.

Ex. 812. *The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.*

Let $A'B'C'$ be a section ∥ to the face ABC of the tetrahedron S-ABC.

To prove that S-$A'B'C'$ is similar to S-ABC.



Proof. $A'B'$, $B'C'$, $C'A'$ are ∥, respectively, to AB, BC, CA. § 528

Hence, the faces $SA'B'$, $SB'C'$, and $SC'A'$ are similar, respectively, to SAB, SBC, and SCA. § 354

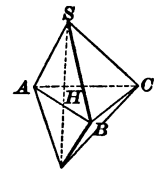
The trihedral $\angle S$ is common to the two tetrahedrons, and the trihedral $\measuredangle A'$, B', and C' are equal to A, B, and C, respectively. § 582

Since the tetrahedrons have their solid $\measuredangle$ equal, and their faces similar and similarly arranged, they are similar. § 664

Q. E. D.

Ex. 813. *Two symmetrical tetrahedrons are equivalent.*

Let $S\text{-}ABC$ and $S'\text{-}ABC$ be two tetrahedrons symmetrical with respect to the plane of their common face ABC.



To prove that $S\text{-}ABC \fallingdotseq S'\text{-}ABC$.

Proof. These tetrahedrons have a common base ABC, and equal altitudes HS and HS'.

$\therefore S\text{-}ABC \fallingdotseq S'\text{-}ABC$. § 650

Q. E. D.

Ex. 814. *Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.*

Let A and B be any two symmetrical polyhedrons.

To prove that they may be decomposed into the same number of tetrahedrons symmetrical each to each.

Proof. Divide A into tetrahedrons by drawing diagonals from any vertex, and divide B into tetrahedrons by drawing in it the homologous diagonals.

Any two corresponding tetrahedrons thus formed have their vertices symmetrical each to each, and are, therefore, symmetrical each to each.

Q. E. D.

Ex. 815. *Two symmetrical polyhedrons are equivalent.*

Let A and B be any two symmetrical polyhedrons.

To prove that $A \fallingdotseq B$.

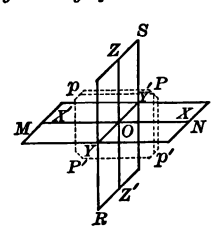
Proof. Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each. Ex. 814

Now any two symmetrical tetrahedrons are equivalent. Ex. 813

$\therefore A \fallingdotseq B$. Q. E. D.

Ex. 816. *If a solid has two planes of symmetry perpendicular to each other, the intersection of these planes is an axis of symmetry of the solid.*

Let MN and RS, intersecting in the line YY', be two perpendicular planes of symmetry of any solid.



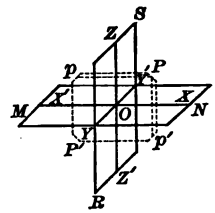
To prove that YY is an axis of symmetry of the solid.

Proof. Through any point P of the surface of the solid pass a plane $\perp$ to the intersection YY'.

This section is $\perp$ to each of the planes MN and RS. § 555

Hence, the points that are the symmetrical points of P, with respect to each plane, lie in this section.

Let p' and p be the symmetrical points of P with respect to the planes MN and RS, respectively, and let XX' and ZZ' be the intersections of the plane of the section with the planes MN and RS.



Then the lines Pp' and Pp are bisected at rt. $\measuredangle$ by XX' and ZZ', respectively.

Hence, the section is symmetrical with respect to the rectangular axes XX' and ZZ'.

Therefore, the section is symmetrical with respect to the point O as a centre of symmetry. § 213

Now the lines joining symmetrical points of this section with respect to O are bisected at rt. $\measuredangle$ by YY'.

Hence, as P is any point in the surface of the given solid, YY' is an axis of symmetry of the solid. § 208

Q. E. D.

Ex. 817. *If a solid has three planes of symmetry perpendicular to each other, the three intersections of these planes are three axes of symmetry of the solid; and the common intersection of these axes is the centre of symmetry of the solid.*

Let a solid be symmetrical with respect to the planes A, B, and C, which are $\perp$ to each other.

To prove that their intersections are axes of symmetry, and that their common point is a centre of symmetry.

Proof. (*i*) From Ex. 816, the intersection of the planes A and B is an axis of symmetry.

For like reason, the intersection of A and C, and that of B and C, are axes of symmetry.

(*ii*) Let P be any point in the surface of the solid.

Construct P', the symmetrical of P with respect to the common point of the planes of symmetry, as a centre of symmetry.

Through P and P' pass planes parallel to each of the planes of symmetry, thus forming a rectangular parallelopiped.

The vertices of this parallelopiped all lie in the surface of the given solid, since any two vertices on the same edge are symmetrical with respect to the plane $\perp$ to that edge, and since P is in the surface.

Hence, any point P of the surface of the solid has its symmetrical point P' in the surface.

Therefore, the intersection of the planes of symmetry is a centre of symmetry. Q. E. D.

Ex. 818. *The volume of a sphere is to the volume of the inscribed cube as* π *is to* $\frac{1}{3}\sqrt{3}$.

Let a denote the edge of the inscribed cube.

Then $2R = a\sqrt{3}$. Ex. 790

$$\therefore a = \frac{2R}{\sqrt{3}} = \tfrac{2}{3}R\sqrt{3}.$$

$\therefore$ the volume of the cube $= (\frac{2}{3}R\sqrt{3})^3 = \frac{8}{9}R^3\sqrt{3}$. § 623

The volume of the sphere $= \frac{4}{3}\pi R^3$. § 845

$\therefore$ sphere : cube $= \frac{4}{3}\pi R^3 : \frac{8}{9}R^3\sqrt{3} = \pi : \frac{2}{3}\sqrt{3}$. Q. E. D.

Ex. 819. *Find the area of the surface of the sphere inscribed in a regular tetrahedron whose edge is* 6 *inches.*

$$R = \frac{a}{12}\sqrt{6} = \tfrac{6}{12}\sqrt{6} = \tfrac{1}{2}\sqrt{6}.$$ Ex. 738

$S = 4\pi R^2 = 4 \times 3.1416 \times (\frac{1}{2}\sqrt{6})^2 = 4 \times 3.1416 \times 1.5 = 18.8496$. § 824

18.85 sq. in. *Ans.*

Ex. 820. *If a zone of one base is the mean proportional between the remainder of the surface of the sphere and the total surface of the sphere, find the distance of the base of the zone from the centre of the sphere.*

Let H = the altitude of the zone.

Then $2R - H$ = the altitude of the zone Z'.

Then $2R - H : H = H : 2R$. § 827

$\therefore 4R^2 - 2RH = H^2$. § 327

$$H^2 + 2RH = 4R^2.$$

$$H^2 + 2RH + R^2 = 5R^2.$$

$$H + R = R\sqrt{5}.$$

$$\therefore H = R(\sqrt{5} - 1).$$

$$\therefore H - R = R(\sqrt{5} - 2).$$

Ex. 821. *Find the difference between the volume of a frustum of a pyramid and the volume of a prism each* 24 *feet high, if the bases of the frustum are squares with sides* 20 *feet and* 16 *feet, respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.*

Volume of frustum $= \frac{1}{3}H(B + b + \sqrt{B \times b})$ § 657

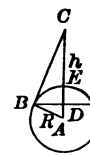
$= \frac{1}{3} \times 24\,(20^2 + 16^2 + \sqrt{20^2 \times 16^2})$

$= \frac{1}{3} \times 24\,(400 + 256 + 320)$
$= \frac{1}{3} \times 24 \times 976$
$= 7808.$

Volume of prism $= B \times H$ § 628
$= [\frac{1}{4}\,(20 + 16)]^2 \times 24$
$= 18^2 \times 24$
$= 7776.$

7808 cu. ft. − 7776 cu. ft. = 32 cu. ft.

Ex. 822. *If the earth is assumed to be a sphere of* 4000 *miles radius, how far at sea can a lighthouse* 100 *feet high be seen?*



100 ft. $= \frac{100}{5280}$ mi. $= \frac{5}{264}$ mi.

$AB = AE = 4000,\ CE = \frac{5}{264}$; $\therefore AC = 4000\frac{5}{264}$.

$BC = \sqrt{\overline{AC}^2 - \overline{AB}^2} = \sqrt{(AC + AB)\,(AC - AB)}$
$= \sqrt{(4000\frac{5}{264} + 4000)\,(4000\frac{5}{264} - 4000)}$
$= \sqrt{8000\frac{5}{264} \times \frac{5}{264}} = \frac{1}{264}\sqrt{2112005 \times 5}$
$= \frac{1}{264}\sqrt{10560025} = \frac{1}{264} \times 3249.6 = 12.3.$

12.3 mi. *Ans.*

Ex. 823. *If the atmosphere extends* 50 *miles above the surface of the earth, and the earth is assumed to be a sphere of* 4000 *miles radius, find the volume of the atmosphere.*

Volume of the earth and atmosphere $= \frac{4}{3}\pi R^3$ § 845
$= \frac{4}{3} \times 3.1416 \times 4050^3 = 4.1888 \times 66{,}430{,}125{,}000.$

Volume of the earth
$= \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.1416 \times 4000^3 = 4.1888 \times 64{,}000{,}000{,}000.$

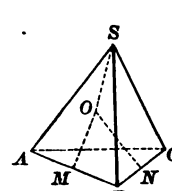
Hence, volume of the atmosphere
$= 4.1888 \times (66{,}430{,}125{,}000 - 64{,}000{,}000{,}000)$
$= 4.1888 \times 2{,}430{,}125{,}000 = 10{,}179{,}307{,}600.$

10,179,307,600 cu. mi. *Ans.*

Ex. 824. *Draw a line through the vertex of any trihedral angle, making equal angles with its edges.*

Let S-ABC be any trihedral $\angle$.

To draw a line through S, making equal $\measuredangle$ with the edges SA, SB, and SC.



Construction. Take SA equal to SB equal to SC, and draw the lines AB, BC, and AC.

In the plane ABC draw the $\perp$s MO and NO to the sides AB and BC at their middle points, respectively.

Connect their common point O with the vertex S.

Then OS is the line required.

Proof. The point O lies in the plane ABC and is equally distant from the three points A, B, and C. § 160

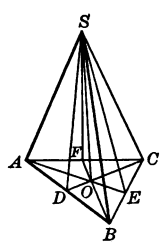
Hence, O is the foot of the $\perp$ from the point S to the plane ABC. § 515

$\therefore$ the $\measuredangle$ OSA, OSB, and OSC are equal. § 515

Q. E. F.

Ex. 825. *In any trihedral angle, the three planes passed through the edges and the respective bisectors of the opposite face angles intersect in the same straight line.*

Let S-ABC be any trihedral $\angle$, and SD, SE, and SF be the bisectors of its face $\measuredangle$.



To prove that the planes ASE, BSF, and CSD intersect in the same straight line.

Proof. Take SA equal to SB equal to SC, and draw the lines AB, BC, and CA.

Draw also AE, BF, and CD, D, E, and F being the points in which SD, SE, and SF cut AB, BC, and CA, respectively.

The $\measuredangle$ SAB, SBC, and SCA are isosceles. Const.

$\therefore$ the bisectors SD, SE, and SF are medians of the respective $\measuredangle$. § 149

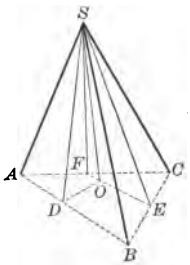
Hence, AE, BF, and CD are the medians of the $\triangle$ ABC.

These medians intersect in some point O. Ex. 27

Hence, the planes ASE, BSF, and CSD have two points O and S in common, and therefore intersect in the straight line SO. Q. E. D.

Ex. 826. *In any trihedral angle, the three planes passed through the bisectors of the face angles, and perpendicular to these faces, respectively, intersect in the same straight line.*

Let S-ABC be any trihedral $\angle$, and SD, SE, and SF the bisectors of its face $\measuredangle$.

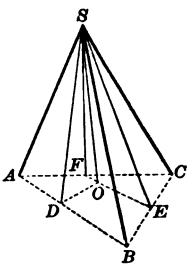


To prove that the planes through SD, SE, and SF, and $\perp$ respectively to the faces ASB, BSC, and CSA, intersect in the same straight line.

Proof. Take SA equal to SB equal to SC, and draw the lines AB, BC, and CA.

Let D, E, and F be their points of intersection with SD, SE, and SF, respectively.

In the $\triangle ABC$, at the points D, E, and F, draw $\perp$s to the sides AB, BC, and CA, respectively.



The $\triangle SAB$, SBC, and SCA are isosceles. Const.

$\therefore$ the bisectors SD, SE, and SF are $\perp$ to AB, BC, and CA, respectively, at their middle points. § 149

Hence, the $\perp$s at D, E, and F, in the plane ABC, intersect in some point O. Ex. 25

Since AB is $\perp$ to DS and DO, it is $\perp$ to the plane SDO. § 507

Hence, the plane ABS is $\perp$ to the plane SDO. § 554

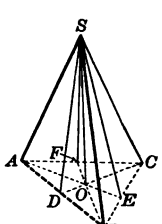
For like reason, the plane OES is $\perp$ to the plane BSC, and the plane OFS to the plane CAS.

Hence, the three planes passed through the bisectors of the face $\measuredangle$s of the trihedral $\angle$, and $\perp$ to these faces, respectively, have the two points O and S in common, and therefore intersect in the straight line OS. Q. E. D.

Ex. 827. *In any trihedral angle, the three planes passed through the edges, perpendicular to the opposite faces, respectively, intersect in the same straight line.*

Let S-ABC be any trihedral $\angle$.

To prove that the three planes passed through the edges SA, SB, and SC, $\perp$ to the opposite faces SBC, SCA, and SAB, respectively, intersect in the same straight line.



Proof. Let the $\perp$ planes through SC and SB intersect in the line SO.

Through SA and SO pass a plane cutting the plane SBC in the line SE.

Through O, any point in SO, pass a plane $\perp$ to SO.

Let AB, BC, and AC be its intersections with the faces of the trihedral $\angle$, and AE, BF, and CD its intersections with the planes through the lateral edges.

Each of the planes SAE, SBF, SCD is $\perp$ to the plane ABC. § 554

Now SCD is $\perp$ to SAB. Hyp.

$\therefore AB$ is $\perp$ to the plane SDC.

Therefore, CD is $\perp$ to AB. § 501

For like reason, BF is $\perp$ to AC.

Hence, in the $\triangle ABC$, BF and CD are lines drawn from the vertices B and C $\perp$ to the opposite sides AC and AB, respectively.

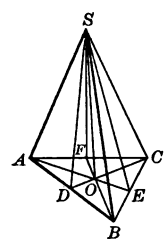
$\therefore$ AE, which is a line from the vertex A through the intersection of the two $\perp$s BF and CD, is $\perp$ to the side BC. Ex. 26

Since the plane SAE is $\perp$ to the plane ABC, and BE is $\perp$ to AE, BE is $\perp$ to the plane AES. § 551

Hence, the plane BSC is $\perp$ to the plane ASE, or ASE is a plane through SA $\perp$ to the opposite face SBC. § 554

Hence, the three planes through the edges SA, SB, and SC, $\perp$ to the opposite faces, respectively, intersect in the straight line SO. Q. E. D.

Ex. 828. *In a tetrahedron, the planes passed through the three lateral edges and the middle points of the opposite sides of the base intersect in a straight line.*



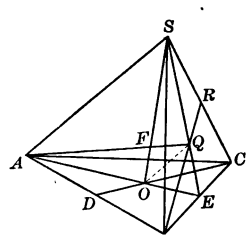
Let S-ABC be any tetrahedron, and D, E, and F the middle points of the sides of its base.

To prove that the planes SAE, SBF, and SCD intersect in the same straight line.

Proof. The lines AE, BF, and CD must intersect in some point O. Ex. 27

Hence, the three planes SAE, SBF, and SCD have two points S and O in common, and therefore intersect in the straight line SO. Q. E. D.

Ex. 829. *The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point, called the centre of gravity, which divides each line so that the shorter segment is to the whole line in the ratio* 1 : 4.



Let S-ABC be any tetrahedron, and AQ and SO the lines drawn from A and S to the intersections Q and O of the medians of the opposite faces BSC and ABC, respectively.

To prove that AQ and SO intersect, and divide each other in the ratio 1 : 4.

Proof. Draw OQ. The lines AQ and SO are drawn in the $\triangle$ AES from the vertices A and S to the opposite sides; hence, they intersect in some point F.

Now

$EO = \frac{1}{3} EA$, and $EQ = \frac{1}{3} ES$. Ex. 27

Hence, the $\triangle$ OEQ and AES are similar. § 357

Therefore, $OQ = \frac{1}{3} AS$, § 351

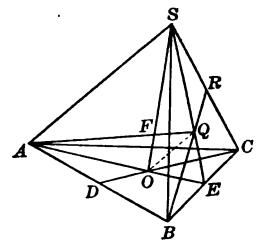
and OQ and AS are parallel. § 345

Hence, the △ OFQ and AFS are mutually equiangular and therefore similar. § 354

$\therefore OF = \frac{1}{3} FS$, and $QF = \frac{1}{3} FA$. § 351

That is, $OF = \frac{1}{2} OS$, and $QF = \frac{1}{2} QA$.

In like manner, it can be shown that the lines drawn from B and C to the intersection of the medians of the opposite faces pass through F and are divided at that point so that the shorter segment is to the whole line in the ratio 1 : 4. Q. E. D.



Ex. 830. *The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the centre of gravity of the tetrahedron, and are bisected by the centre of gravity.*

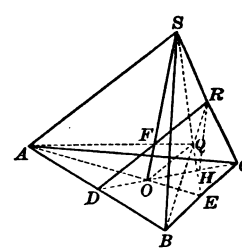
Let S-ABC be any tetrahedron in which DR connects the middle points of the opposite edges AB and SC.

To prove that DR is bisected by the centre of gravity of the tetrahedron.

Proof. Let O be the intersection of the medians AE and CD of the face ABC.

Draw SO, and through R draw $RH \parallel$ to SO.

Since SO and RD lie in the plane DCS, they evidently intersect in some point F.



Now $HC = OH$, and $RH = \frac{1}{2} SO$. § 342

But $DO = \frac{1}{2} OC$. Ex. 27

Hence, $DO = OH$.

Whence, $OF = \frac{1}{3} HR = \frac{1}{3} SO$, and $DF = FR$.

For the △ OFD and HRD are similar, and $OD = \frac{1}{3} DH$.

That is, F is the centre of gravity of the tetrahedron and bisects DR.

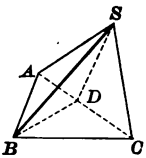
In like manner, we may prove that the line joining the middle points of any other two opposite edges is bisected by the centre of gravity.

Q. E. D.

Ex. 831. *The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the areas of the faces that include the dihedral angle.*

Let the plane SBD bisect the dihedral $\angle$ A-SB-C of the trihedral $\angle$ S-ABC.

To prove that $\dfrac{AD}{DC} = \dfrac{\triangle SAB}{\triangle SBC}$.



Proof. Since the tetrahedrons $S\text{-}ABD$ and $S\text{-}DBC$ have the same altitude, they are to each other as their bases. § 653

That is, $\dfrac{S\text{-}ABD}{S\text{-}DBC} = \dfrac{\triangle DBA}{\triangle DBC}$.

Since the $\triangle DBA$ and DBC have the same altitude, they are to each other as their bases. § 405

That is, $\dfrac{\triangle DBA}{\triangle DBC} = \dfrac{DA}{DC}$.

$\therefore \dfrac{S\text{-}ABD}{S\text{-}DBC} = \dfrac{DA}{DC}$. Ax. 1

Consider D as the common vertex of the tetrahedrons $S\text{-}ABD$ and $S\text{-}DBC$, and SAB and SBC as their bases, respectively.

Then their altitudes are equal. § 559

Hence, $\dfrac{D\text{-}SAB}{D\text{-}SBC} = \dfrac{\triangle SAB}{\triangle SBC}$, or $\dfrac{S\text{-}ABD}{S\text{-}DBC} = \dfrac{\triangle SAB}{\triangle SBC}$. § 653

$\therefore \dfrac{DA}{DC} = \dfrac{\triangle SAB}{\triangle SBC}$. Q. E. D.

Ex. 832. *The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within the tetrahedron upon the four faces.*

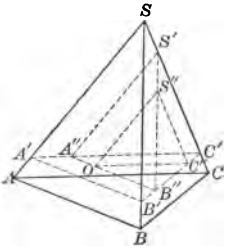
Let O be any point within the regular tetrahedron $S\text{-}ABC$.

To prove that the sum of the four $\perp$s from O upon the faces of the tetrahedron is equal to the altitude of the tetrahedron.

Proof. Through O pass the plane $A'B'C' \parallel$ to the face ABC.

Then $S\text{-}A'B'C'$ is another regular tetrahedron. Ex. 812

The altitude of $S\text{-}ABC$ is equal to that of $S\text{-}A'B'C'$ plus the $\perp$ from O upon the face ABC.



Through O pass the plane $S'A''B'' \parallel$ to the face SAB.

Then $S'\text{-}A''B''C'$, or $C'\text{-}S'A''B''$, is a regular tetrahedron. Ex. 812

The altitude of $C'\text{-}SA'B'$ is equal to that of $C'\text{-}S'A''B''$ plus the $\perp$ from O upon the face SAB.

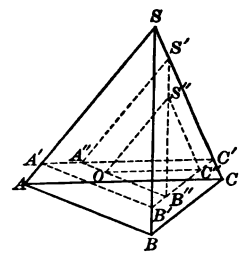
Again through O pass the plane $OS''C''$ ∥ to the face SAC.

Then B''-$OS''C''$ is a regular tetrahedron.

Ex. 812

The altitude of B''-$A''S'C'$ is equal to that of B''-$OS''C''$ plus the ⊥ from O upon the face SAC, and the altitude of B''-$OS''C''$ is equal to the ⊥ from O upon the face SBC.

Hence, the altitude of S-ABC equals the sum of the four ⊥s from O upon the four faces of the tetrahedron.



Q. E. D.

Ex. 833. *Within a given tetrahedron, to find a point such that the planes passed through this point and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.*

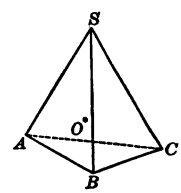
Let S-ABC be any tetrahedron.

To find a point within the tetrahedron such that the planes passed through it and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.

Construction. Construct a plane ∥ to ABC at a distance equal to one fourth the distance of S from ABC.

Construct a second plane ∥ to SAB at a distance equal to one fourth the distance of C from SAB.

Construct a third plane ∥ to SAC at a distance equal to one fourth the distance of B from SAC.



Let O be the intersection of these three planes.

Then O is the point required.

Proof. By construction this point O is within the tetrahedron.

The pyramid having ABC as base and O as vertex $= \frac{1}{4}$ S-ABC, since it has the same base and one fourth the altitude of S-ABC. § 653

For like reason, $O\text{-}SAB = \frac{1}{4}$ $C\text{-}SAB = \frac{1}{4}$ $S\text{-}ABC$,

and $O\text{-}SAC = \frac{1}{4}$ $B\text{-}SAC = \frac{1}{4}$ $S\text{-}ABC$.

$\therefore$ $O\text{-}SBC = \frac{1}{4}$ $S\text{-}ABC$.

$\therefore$ $O\text{-}ABC = O\text{-}SAB = O\text{-}SAC = O\text{-}SBC$. Q. E. F.

Ex. 834. *To cut a cube by a plane so that the section shall be a regular hexagon.*

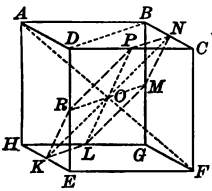
Let A-$HEFG$ be any cube.

To cut the cube by a plane so that the section shall be a regular hexagon.

Construction. Draw AF, the diagonal of the cube, and through its middle point O pass a plane $\perp$ to it, forming the section $KLMNPR$.

The section $KLMNPR$ is the regular hexagon required.

Proof. The point P is equally distant from A and F. § 517



But $\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$,

and $\overline{FP}^2 = \overline{FC}^2 + \overline{CP}^2$. § 371

Hence, $\overline{DP}^2 = \overline{CP}^2$, or $DP = CP$.

That is, P is the middle point of the edge DC.

For like reason, N is the middle point of BC, etc.

Hence, $PN = \frac{1}{2}\, DB$. § 189

Or each side of the section is equal to one half the diagonal of a face.

The section through O, $\parallel$ to $DAHE$, is equal to $DAHE$ and passes through P and L.

Hence, $OP = OL = \frac{1}{2}$ diagonal DH.

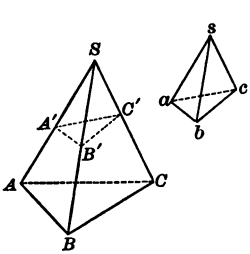
Hence, the $\triangle$ OPR, ORK, etc., are equilateral $\triangle$, and, therefore, the $\angle$ RPN, PNM, etc., are each equal to 120°.

Hence, the section is a regular hexagon. § 429

Q. E. F.

Ex. 835. *Two tetrahedrons are similar if a dihedral angle of one is equal to a dihedral angle of the other, and the faces that include these angles are respectively similar, and similarly placed.*

Let S-ABC and s-abc be two tetrahedrons having the dihedral $\angle SB$ equal to the dihedral $\angle sb$, and the faces SAB and SBC similar, respectively, to sab and sbc.



To prove that S-ABC and s-abc are similar.

Proof. Place the tetrahedron s-abc on S-ABC so that their vertices and equal dihedral $\angle$ coincide.

Then, since the $\angle BSA =$ the $\angle bsa$, and the $\angle BSC = \angle bsc$ (§ 351), sc will fall on SC, and sa on SA.

Let S-$A'B'C'$ denote the position of s-abc.

Then $A'B'$ and $B'C'$ are $\parallel$ to AB and BC, respectively, since the faces SAB and SBC are similar to sab and sbc, respectively.

Hence, $A'B'C'$ is a section $\parallel$ to the face ABC. § 534

Therefore, S-$A'B'C'$, or s-abc, is similar to S-ABC. Ex. 812

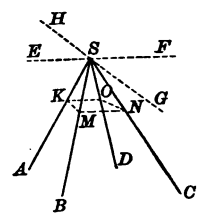
Q. E. D.

Ex. 836. *To cut a tetrahedral angle so that the section shall be a parallelogram.*

Let $S\text{-}ABCD$ be any tetrahedral $\angle$.

To cut $S\text{-}ABCD$ by a plane so that the section shall be a $\square$.

Construction. Let ESF be the intersection of the planes of the opposite faces BSC and ASD, and HSG that of the planes of the opposite faces ASB and CSD.



These two lines of intersection determine a plane HSE through the vertex S. § 497

Through K, any point in the edge SA, pass a plane $\parallel$ to the plane HSE, forming the section $KMNO$.

Then $KMNO$ is the $\square$ required.

Proof. The lines KM and HG are parallel. § 528

For like reason, ON and HG are parallel.

Hence, KM and ON are parallel. § 521

Similarly, MN and KO are parallel.

Hence, $KMNO$ is a $\square$. § 166

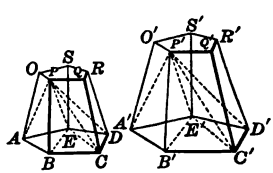
Q. E. F.

Discussion. The cutting plane may be made to pass through any point on the side of the plane ESH on which the tetrahedral angle lies.

Thus, there is an indefinite number of solutions of the problem.

Ex. 837. *Two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed, are similar.*

Let OD and $O'D'$ be two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed.



To prove that OD and $O'D'$ are similar.

Proof. Any two corresponding solid $\measuredangle$s, as B and B', are made up of equal solid $\measuredangle$s of the similar tetrahedrons similarly placed.

Hence, both the solid $\measuredangle$ and the face $\measuredangle$ are equal respectively.

Moreover, from the similar tetrahedrons we have

$$\frac{AB}{A'B'} = \frac{PB}{P'B'} = \frac{BC}{B'C'} = \text{etc.}\,; \qquad \text{§ 667}$$

that is, the homologous edges are proportional.

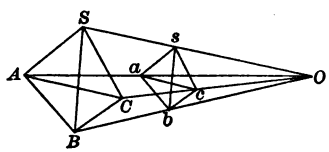
Hence, the corresponding solid $\measuredangle$ being equal, and the corresponding faces similar, the polyhedrons are similar. § 664

Q. E. D.

Ex. 838. *If the homologous faces of two similar pyramids are respectively parallel, the straight lines which join the homologous vertices of the pyramids meet in a point.*

Let the homologous faces of the two similar pyramids $S\text{-}ABC$ and $s\text{-}abc$ be ||, respectively.

To prove that the lines Ss, Aa, Bb, and Cc through the homologous vertices meet in some point O.



Proof. Since the homologous faces are parallel, the homologous edges are also parallel.

Let O be the point in which Aa meets Ss.

Then the $\triangle OSA$ and Osa, are similar. § 354

Hence, $OS : Os = AS : as.$ § 351

$\therefore OS - Os : SO = AS - as : AS.$ § 333

That is, $Ss : SO = AS - as : AS.$

Again, let O' be the point in which Cc cuts Ss produced.

Then $O'S : O's = CS : cs = AS : as.$

$\therefore Ss : SO' = AS - as : AS.$ § 333

$\therefore SO' = SO.$

That is, Cc cuts Ss in the same point that Aa cuts Ss.

In like manner, it may be shown that Bb cuts Ss in the same point that Cc cuts Ss.

Hence, all the lines Ss, Aa, Bb, and Cc meet in a point. Q. E. D.

Ex. 839. *The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.*

$S = 2\pi RH.$ § 698

$V = \pi R^2 H = \frac{1}{2} R(2\pi RH) = \frac{1}{2} RS.$ § 700

Q. E. D.

Ex. 840. *The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.*

Let R be the radius, H the altitude, and V the volume of any right circular cylinder.

Then R is the base and H the altitude of the generating rectangle.

Since the diagonals of a $\square$ intersect midway between its parallel sides, $\frac{1}{2} R$ is the radius of the circumference generated by the point of intersection of the diagonals.

Now the area of rectangle $= RH$. § 398

The circumference described by the intersection of the diagonals equals πR. § 458

$$V = \pi R^2 H = (RH)(\pi R). \quad \text{§ 700}$$

That is, the volume is equal to the product of the area of the generating rectangle by the length of the circumference generated by the point of intersection of its diagonals. Q. E. D.

Ex. 841. *If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to total area multiplied by a third of the radius.*

Let R be the radius of the base of a right circular cylinder, $2R$ the altitude, V the volume, and T the total area.

To prove that $V = \frac{1}{3} RT$.

Here $T = 2\pi R^2 + (2\pi R)\, 2R = 6\pi R^2$, § 698

and $V = (\pi R^2)\, 2R = \frac{1}{3} R\,(6\pi R^2) = \frac{1}{3} RT$. § 700

Q. E. D.

Ex. 842. *Show that the prismatoid formula can be used for finding the volume of a sphere.*

$$V = \tfrac{1}{6} H(B + b + 4M). \quad \text{§ 733}$$

In a sphere B and $b = 0$, and M is a great circle.

$$\therefore 4M = 4\pi R^2 \quad \text{§ 463}$$

$$\therefore V = \tfrac{1}{6} \times 2R \times 4\pi R^2 = \tfrac{4}{3}\pi R^3. \quad \text{§ 845}$$

Ex. 843. *Find the altitude of a zone equivalent to a great circle.*

$$Z = 2\pi RH. \quad \text{§ 826}$$

$$\text{Great } \odot = \pi R^2. \quad \text{§ 463}$$

$$\therefore 2\pi RH = \pi R^2.$$

$$2H = R.$$

$$\therefore H = \tfrac{1}{2} R.$$

Ex. 844. *Find the area of a spherical pentagon whose angles are* 122°, 128°, 131°, 160°, 161°, *if the surface of the sphere is* 150 *square feet.*

$$E = T - (n - 2)\,180°.$$

$$T = 122° + 128° + 131° + 160° + 161° = 702°.$$

$$E = 702° - 3 \times 180° = 162°.$$

$\therefore$ area pentagon $= \frac{162}{720}$ of 150 sq. ft. $= 33\frac{3}{4}$ sq. ft.

Ex. 845. *Construct a spherical surface with a given radius passing through a given point and tangent to two given planes.*

Let R be any given radius, A any given point, and MN and PQ any two given planes.

To pass a spherical surface with radius R through the point A, tangent to the planes MN and PQ.

Analysis. The centre of the sphere must be in the surface of the sphere having A as centre and R as radius. It must also be in the plane $\parallel$ to the plane MN at the distance R, on the side of which A lies, and in the plane $\parallel$ to the plane PQ at the distance R on the A side.

Construction. Construct the sphere with A as centre and R as radius. Construct the plane $\parallel$ to the plane MN at the distance R on the A side; also the plane $\parallel$ to the plane PQ at the distance R on the A side.

The spheres having as centres the points common to the intersection of these constructed planes and the surface of the constructed sphere, and the given radius R will pass through the point A and be tangent to each of the planes MN and PQ.

Proof. The surface of each of these spheres will pass through the point A, since by construction A is at the distance R from its centre.

Each of these spheres will be tangent to each of the planes MN and PQ, since by construction R is the perpendicular distance from its centre to each of the planes MN and PQ. Q. E. F.

Discussion. (*i*) When the two given planes are not parallel: (1) If the given point lies in either of the dihedral $\measuredangle$ formed by the given planes, there will be two, one, or no solution, according as the surface of the constructed sphere has two, one, or no points in common with the intersection of the constructed planes. (2) If the given point is in either given plane, there will be a solution if the $\perp$ to the given plane at the given point cuts either intersection of the constructed planes at the distance R from the given point. Otherwise there will be no solution.

(*ii*) When the two given planes are parallel: If the distance between the two planes is $2R$, and the given point lies between them, there will be an unlimited number of spheres fulfilling the given conditions; that is, the problem will be indeterminate. If the given point lies in either of the parallel planes, there will be one, and only one, solution. There will be no solution when the point does not lie between the planes, or in one of them, or when the distance between the planes is not $2R$.

Ex. 846. *Construct a spherical surface with a given radius passing through a given point and tangent to two given spheres.*

Let R be any given radius, A any given point, and O and O' the centres of any two given spheres whose radii are R' and R'', respectively.

To pass a spherical surface with radius R through the point A, tangent to the spheres O and O'.

Case I. *Point without both spheres, neither of which is within the other.*

Construction. Construct the sphere with A as centre and R as radius. Construct also the two spheres with O and O' as centres and $R' + R$ and $R'' + R$ as radii, respectively.

The spheres having as centres the points common to the surfaces of these three constructed spheres, and the given radius R will pass through the point A and be tangent to each of the spheres O and O'.

Proof. The surface of each of these spheres will pass through the point A, since by construction A is at the distance R from its centre.

Each of these spheres will be tangent to each of the spheres O and O', since by construction the distances of its centre from O and O' are $R' + R$ and $R'' + R$, respectively.

CASE II. *Point and one sphere within the other sphere.*

Suppose the point A and the sphere O are within the sphere O'.

Construction. Construct the sphere with A as centre and R as radius. Construct also the two spheres with O and O' as centres and $R' + R$ and $R'' - R$ as radii, respectively.

The spheres having as centres the points common to the surfaces of these three constructed spheres, and the given radius will pass through the point A and be tangent to each of the spheres O and O'.

The proof is similar to that in Case I.

CASE III. *Point in surface of one sphere.*

This will fall under Case I or II, according as the second sphere lies without or within the first. Q. E. F.

Discussion. If the point lies within one sphere, and the other sphere lies without, or if one sphere lies within the other and the point lies without, there will be no solution.

EX. 847. *Construct a spherical surface with a given radius passing through a given point and tangent to a given plane and a given sphere.*

Let R be any given radius, A any given point, MN any given plane, and O the centre of any given sphere whose radius is R'.

To pass a spherical surface with radius R through the point A, tangent to the plane MN and the sphere O.

Construction. Construct the sphere with A as centre and R as radius. Construct also the spheres with O as centre and $R' + R$ and $R' - R$ as radii, respectively; and construct the planes $\parallel$ to the given plane at the distance R from it.

The spheres having as centres the points common to the sphere A, either of the constructed parallel planes, and either of the spheres constructed with O as centre, and the given radius R, will pass through the point A, be tangent to the given sphere O and the given plane MN.

Proof. The surface of each of these spheres will pass through the point A, since by construction A is at the distance R from its centre.

Each of these spheres will be tangent to the plane MN, since by construction R is the perpendicular distance from its centre to the plane MN.

Each will be tangent to the given sphere O, since by construction the distance of its centre from O is $R' + R$ or $R' - R$. Q. E. F.

Ex. 848. *Construct a spherical surface with a given radius tangent to three given planes.*

Let R be any given radius, and MN, PQ, and RS any three given planes.

To construct a spherical surface with radius R, tangent to the three planes MN, PQ, and RS.

Construction. Construct the two planes $\parallel$ to the plane MN at the distance R from it. Construct also the planes $\parallel$ to the planes PQ and RS, respectively, at the distance R from each.

The spheres having as centres the points common to any three of the six constructed planes, and the given radius R, will be tangent to each of the three given planes.

Proof. Each of these spheres will be tangent to each of the given planes, since by construction R is the perpendicular distance from its centre to each of the planes MN, PQ, and RS. Q. E. F.

Discussion. (*i*) If the given planes have but a single point in common, there will be eight different spheres.

(*ii*) If the given planes have a line in common, or if they are parallel, there will be no solution.

Ex. 849. *Construct a spherical surface with a given radius tangent to three given spheres.*

Let R be any given radius, and O_1, O_2, and O_3 the centres of any three given spheres whose radii are R_1, R_2, and R_3, respectively.

To construct a spherical surface with radius R tangent to the three given spheres O_1, O_2, and O_3.

Construction. Construct the sphere with O_1 as centre and $R_1 + R$ as radius; construct also the spheres with O_2 and O_3 as centres and $R_2 + R$ and $R_3 + R$ as radii, respectively.

The spheres having as centres the points common to the surfaces of these three constructed spheres and the given radius R will be tangent to each of the three given spheres O_1, O_2, and O_3.

Proof. Each of these spheres will be tangent to each of the given spheres, for the distance between the centre of each and each of the given spheres is equal to the sum of their radii. Q. E. F.

Discussion. There will be two, one, or no solution, according as the surfaces of the three constructed spheres have two, one, or no points in common.

If the three constructed spheres intersect in the same ⊙, the problem is indeterminate.

If one of the given spheres lies within, and another without, the third, the problem is impossible.

If two of the given spheres lie within the third O_3, construct the spheres with O_1, O_2, and O_3 as centres and $R_1 + R$, $R_2 + R$, and $R_3 - R$ as radii, respectively. The points common to the surfaces of these three spheres will be the centres of the required spheres.

Ex. 850. *Construct a spherical surface with a given radius tangent to two given planes and a given sphere.*

Let R be any given radius, MN and PQ any two given planes, and O the centre of a sphere whose radius is R'.

To construct a spherical surface with radius R tangent to each of the planes MN and PQ, and to the sphere O.

Construction. Construct the spheres with O as a common centre, and $R' + R$ and $R' \smile R$ as radii, respectively. Construct the two planes ∥ to the plane MN at the distance R; also construct the two planes ∥ to the plane PQ at the distance R.

The spheres having as centres the points common to either two of these constructed planes and either one of the constructed spheres, and the given radius R will be tangent to each of the planes MN and PQ and to the sphere O.

Proof. Each of these spheres will be tangent to each of the planes MN and PQ, since by construction R is the perpendicular distance from its centre to each of these planes.

Each will be tangent to the sphere O, since the distance between its centre and the centre O is $R' + R$ or $R' \smile R$. Q. E. F.

Discussion. (*i*) When the two given planes intersect. If the intersection of the two planes passes through the given sphere, there *may* be as many as sixteen different spheres constructed. If both planes do not cut the sphere, there may be two, one, or no solution.

(*ii*) When the two given planes are parallel. If the distance between the planes is $2R$, the problem is indeterminate, has one or no solution, according as the plane midway between the given planes cuts, is tangent to, or has no point in common with the surface of the sphere whose radius is $R' + R$. If the distance between the planes is greater or less than $2R$, there will be no solution.

Ex. 851. *Construct a spherical surface with a given radius tangent to two given spheres and a given plane.*

Let R be any given radius, MN any given plane, and O_1 and O_2 the centres of two spheres whose radii are R_1 and R_2, respectively.

To construct a spherical surface with radius R tangent to the plane MN and to the spheres O_1 and O_2.

Construction. Construct the two planes $\parallel$ to the plane MN at the distance R; also construct the spheres with O_1 and O_2 as centres and $R_1 + R$ and $R_2 + R$ as radii, respectively.

The spheres having as centres the points common to either of the two planes and both the constructed spheres, and the given radius R, will be tangent to the plane MN and to the spheres O_1 and O_2.

Proof. Each of these spheres will be tangent to the plane MN, since by construction R is the perpendicular distance of its centre from this plane.

Each will be tangent to the spheres O_1 and O_2, since the distances of its centre from O_1 and O_2 are $R_1 + R$ and $R_2 + R$, respectively. Q. E. F.

Discussion. If the given spheres are not on opposite sides of the given plane, there may be two, one, or no solutions.

If the given spheres are on opposite sides of the given plane, the problem is impossible, except when they are tangent to the given plane at the same point, when there are two solutions.

If one sphere is within the other, the problem is impossible, except when the given plane is tangent to or cuts the greater sphere.

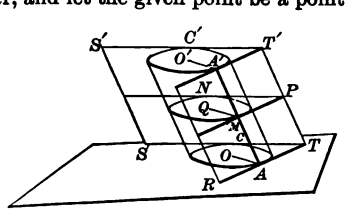
If the intersection of the constructed spheres O_1 and O_2 coincides with either plane $\parallel$ to MN, the problem is indeterminate.

Ex. 852. *Through a given point to pass a plane tangent to a given circular cylinder.*

Case I. *When the given point is in the curved surface of the cylinder.*

Let AC' be a given circular cylinder, and let the given point be a point in the element AA'.

To pass a plane tangent to the cylinder, embracing the element AA'.



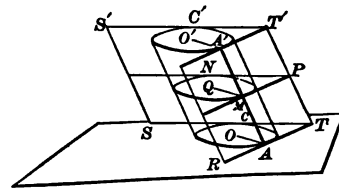
Construction. Draw the radius OA, and AT tangent to the base.

Pass a plane RT' through AA' and AT.

The plane RT' is the plane required.

Proof. Through any point P in this plane, not in the element AA', pass a plane ∥ to the base, intersecting the cylinder in the ⊙ MN, and the plane RT' in MP. §§ 694, 506

From the centre of the ⊙ MN draw QM.



MP and MQ are ∥, respectively, to AT and OA. § 528

$\therefore \angle PMQ = \angle TAO$. § 534

Therefore, PM is tangent to the ⊙ MN at M. § 253

Therefore, P lies without the ⊙ MN, and hence without the cylinder.

Therefore, the plane RT' contains the element AA', but does not cut the cylinder, and is therefore tangent to the cylinder. § 686

Case II. *When the given point is without the cylinder.*

Let P be the given point.

To pass a plane through P tangent to the cylinder.

Construction. Through P draw the line PT ∥ to the elements of the cylinder, meeting the plane of the base at T.

From T draw TA and TC tangents to the base. § 317

The planes ATP and CTP are the required tangent planes.

Proof. Since AA' is ∥ to PT (Const.), the plane RT', passing through PT and the point A, will contain the element AA'.

Since RT' also contains the tangent AT, it is a tangent plane to the cylinder. Case I

In like manner, the plane TS' is a tangent plane to the cylinder. Q. E. F.

Ex. 853. *Through a given point to pass a plane tangent to a given circular cone.*

Case I. *When the given point is in the curved surface of the cone.*

Let S-BD be a given circular cone, and let the given point be a point in the element BS.

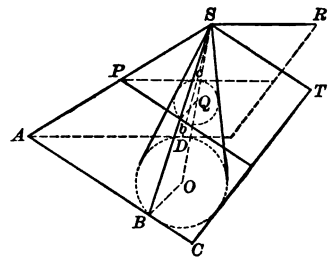
To pass a plane tangent to the cone embracing the element BS.

Construction. Draw the radius OB, and the tangent AB to the base at B, and pass a plane through BS and AB.

Then the plane ABS is the tangent plane required.

Proof. Through any point P in this plane, not in the element BS, pass a plane ∥ to the base, intersecting the cone in the ⊙ bc, and the plane ABS in Pb. §§ 718, 506

From the centre Q of the ⊙ bc draw Qb.



Pb and Qb are ||, respectively, to AB and OB. § 528

$\therefore \angle PbQ = \angle ABO$. § 534

Hence, Pb is tangent to the $\odot bc$ at b. § 253

Therefore, P lies without the $\odot bc$, and hence without the cone.

Hence, the plane ABS contains the element BS, but does not cut the cone, and is therefore tangent to the cone. § 709

CASE II. *When the given point is without the cone.*

Let P be the given point without the cone S-BD.

To pass a plane through P tangent to the cone.

Construction. Draw SP, and produce it to meet the plane of the base in some point A.

From A draw AB and AD tangents to the base, and through SA and these tangents pass the planes ABS and ADS.

Then ABS and ADS are the tangent planes required.

Proof. Since the lines SB and SD are elements of the cone, each of the planes ABS and ADS is the plane of an element and a tangent to the base at the foot of the element.

Hence, by the proof in Case I, each of these planes is tangent to the cone. Q. E. F.

Scholium. When the given point is without the cone the problem may be stated as follows: Through a given line passing through the vertex of a cone to pass a plane tangent to the cone.

Ex. 854. *Find the radius and the surface of a sphere whose volume is one cubic yard.*

$V = \frac{4}{3}\pi R^3$. § 845

$\therefore R^3 = \frac{3V}{4\pi}$.

$$\therefore R = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3}{4\pi}} = \sqrt[3]{0.75 \times 0.31831} = \sqrt[3]{0.2387325} = 0.62035.$$

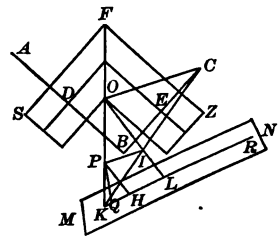
$S = 4\pi R^2 = 4 \times 3.1416 \times 0.62035^2 = 4.836$. § 824

$\therefore R = 0.62035$ yd.; $S = 4.836$ sq. yd.

Ex. 855. *Find the centre of a sphere whose surface passes through three given points and touches a given plane.*

Let A, B, and C be any three given points in space, and MN any given plane.

To find the centre of a sphere whose surface shall pass through A, B, and C, and be tangent to the plane MN.



Analysis. The centre must be equally distant from A and B; hence, it must be in the plane $\perp$ to the line AB at its middle point. § 517

For like reason, it must be in the plane $\perp$ to BC at its middle point.

Therefore, the centre is in the line of intersection of these two planes.

Again, the centre must be equally distant from the point C and the plane MN.

Hence, it is the point in this intersection that is equally distant from the point C and the plane MN.

Construction. Draw AB and BC, and bisect each of them.

Through the middle points D and E pass the planes SOF and ZOF $\perp$ to AB and BC, respectively.

Let FP be the intersection of these planes, which meets MN in K.

Through FP pass a plane $\perp$ to the plane MN, intersecting it in KR.

Draw CK, and from any point P in KF draw $PH \perp$ to KR.

With P as centre and radius equal to PH, strike an arc in plane FKC, cutting KC in the points I and Q.

Draw PI and through C draw $CO \parallel$ to IP, intersecting FP at O.

Then O is the centre of the spherical surface through A, B, and C, and tangent to the plane MN, and OC is its radius.

Proof. The point O is equidistant from A, B, and C. § 517

From O draw $OL \perp$ to KR.

Then OL is $\perp$ to the plane MN. § 551

The lines PH and OL are parallel. § 104

Hence, the $\triangle KPH$ and KOL are similar. § 354

$\therefore KP : KO = PH : OL.$ § 351

The $\triangle KPI$ and KOC are also similar. § 354

$\therefore KP : KO = PI : OC$ § 351

$\therefore PH : OL = PI : OC.$ Ax. 1

But $PH = PI.$ Const.

$\therefore OL = OC.$

Hence, the spherical surface with centre O, and with radius OL or OC, will pass through the three points A, B, and C, and be tangent to the plane MN. Q. E. F.

Discussion. There is no solution if two points are in the plane MN or on opposite sides of it.

When one point, as A, is in the plane, there is one or no solution, according as the line $\perp$ to MN at A cuts or does not cut the intersection of the planes $\perp$ to AB and BC at their middle points.

When the three points are on the same side of the plane there will be two solutions, one solution, or no solution, according as the $\angle CKF$ is less than, equal to, or greater than the $\angle FKR$.

Ex. 856. *Find the centre of a sphere whose surface touches two given planes and passes through two given points which lie between the planes.*

Let A and B be any two points between the planes MN and PQ.

To find the centre of a spherical surface which shall pass through the points A and B and be tangent to the planes MN and PQ.

Analysis. The centre must be equidistant from A and B, and, therefore, in the plane $\perp$ to AB at its middle point.

The centre must be equidistant from the planes MN and PQ, and, therefore, in the plane bisecting the dihedral $\angle$ included between them.

Hence, the centre is in the line of intersection of these two planes.

Again, the centre must be equidistant from the point B and the plane MN.

Hence, it is the point in this line of intersection that is equidistant from the point B and the plane MN.

Construction. Bisect the dihedral $\angle$ between the planes MN and PQ, draw AB, and bisect it with a perpendicular plane.

Let FK be the intersection of this plane with the plane bisecting the dihedral $\angle$ between the given planes.

Find, as in Ex. 855, the point O in FK at equal distances from B and the plane MN.

Then O is the centre of the spherical surface through the points A and B, tangent to the planes MN and PQ.

Proof. Since O is in the plane bisecting the $\angle$ included between the given planes, it is equidistant from these planes. § 559

Since O is in the plane $\perp$ to AB at its middle point, it is equidistant from A and B. § 517

O is equidistant from the point B and the plane MN. Const.

Hence, O is equidistant from the points A and B and the planes MN and PQ, and is, therefore, the centre of the spherical surface which passes through A and B and is tangent to the planes MN and PQ. Q. E. F.

Discussion. If the given planes are parallel, instead of the bisecting plane, we draw a plane midway between the parallel planes.

As in Ex. 855, when the points are both on the same side of the plane, there may be two solutions, one solution, or no solution.

Ex. 857. *The volume of a sphere is two thirds the volume of the circumscribed cylinder, and its surface is two thirds of the total surface of the cylinder.*

Let V denote the volume, and S the surface of any sphere; and let V' denote the volume, and T the total surface of the circumscribed cylinder.

To prove that $V = \frac{2}{3} V'$, and $S = \frac{2}{3} T$.

Proof. Now $H = 2R$.

$V = \frac{4}{3} \pi R^3$. § 845

$V' = \pi R^2 H = 2 \pi R^3$. § 700

$\therefore \frac{V}{V'} = \frac{\frac{4}{3} \pi R^3}{2 \pi R^3} = \frac{2}{3}$.

$\therefore V = \frac{2}{3} V'$.

Again, $S = 4 \pi R^2$. § 824

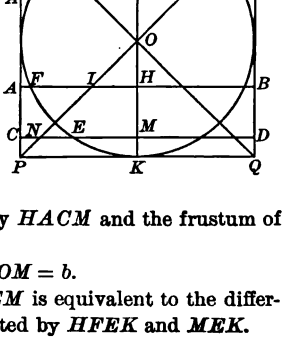
$T = 2 \pi R (H + R) = 6 \pi R^2$. § 698

$\therefore \frac{S}{T} = \frac{4 \pi R^2}{6 \pi R^2} = \frac{2}{3}$.

$\therefore S = \frac{2}{3} T$. Q. E. D.

Ex. 858. *Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindrical and conic segments.*

Let LFK be any $\odot$, $RGPQ$ a circumscribed square, RP and GQ its diagonals, LK a diameter of the $\odot$ $\parallel$ to RQ, and AB and CD any two lines cutting the $\odot$ $\perp$ to LK.



Then, if the figure is revolved about LK as an axis, the $\odot$ will generate a sphere, the square the circumscribed cylinder, the $\triangle$ ORG and OPQ the inscribed cone of two nappes, and AB and CD the planes $\perp$ to the axis LK.

To prove that, when the figure is revolved about LK, the spherical segment generated by $HFEM$ is equivalent to the difference between the cylinder generated by $HACM$ and the frustum of a cone generated by $HINM$.

Proof. Let $OK = R$, $OH = a$, $OM = b$.

The spherical segment generated by $HFEM$ is equivalent to the difference between the spherical segments generated by $HFEK$ and MEK.

If h denotes the altitude of a spherical segment of one base, from § 849 we have for the volume of a spherical segment of one base

$$V = \pi h^2 (R - \tfrac{1}{3} h).$$

If
$$h = HK = R - a,$$
$$V = \pi (R - a)^2 [R - \tfrac{1}{3} (R - a)]$$
$$= \pi (\tfrac{2}{3} R^3 - aR^2 + \tfrac{1}{3} a^3). \quad (1)$$

If
$$h = MK = R - b,$$
$$V = \pi (R - b)^2 [R - \tfrac{1}{3} (R - b)]$$
$$= \pi (\tfrac{2}{3} R^3 - bR^2 + \tfrac{1}{3} b^3). \quad (2)$$

Subtract (2) from (1).

Then volume $HFEM = \pi R^2 (b - a) - \tfrac{1}{3} \pi (b^3 - a^3)$. (3)

Since $MC = R$, and $MH = OM - OH = b - a$, we have, by § 700,

volume $HACM = \pi R^2 (b - a)$. (4)

Since $PK = OK = R$, $NM = MO = b$, and $IH = OH = a$, the volume generated by $HINM$ is equal to the cone generated by OMN diminished by the cone generated by OHI.

Cone $OMN = \tfrac{1}{3} MO \times \pi \overline{NM}^2 = \tfrac{1}{3} \pi b^3$. § 725

Cone $OHI = \tfrac{1}{3} OH \times \pi \overline{IH}^2 = \tfrac{1}{3} \pi a^3$.

$\therefore$ volume $HINM = \tfrac{1}{3} \pi (b^3 - a^3)$. (5)

From (3), (4), and (5) we obtain

volume $HFEM$ = volume $HACM$ − volume $HINM$.

When the planes are on opposite sides of the centre O, as those generated by $A'B'$ and CD, the volume of the spherical segment generated by $H'F'EM$ is equal to the volume of the entire sphere diminished by the volume of the segments generated by EMK and $F'H'L$, respectively.

Hence, if $OH' = a$,

volume $H'F'EM = \tfrac{4}{3} \pi R^3 - \pi (\tfrac{2}{3} R^3 - aR^2 + \tfrac{1}{3} a^3) - \pi (\tfrac{2}{3} R^3 - bR^2 + \tfrac{1}{3} b^3)$
$= \pi R^2 (a + b) - \tfrac{1}{3} \pi (a^3 + b^3)$
$=$ volume $MH'A'C$ − volume $OH'I'$ − volume OMN.

Q. E. D.

Ex. 859. *A sphere* 12 *inches in diameter has an auger hole* 3 *inches in diameter through its centre. Find the remaining volume.*

The volume of the sphere $= \tfrac{1}{6} \pi D^3 = \tfrac{1}{6} \times 3.1416 \times 12^3 = 904.7808$.

§ 845

The hole is composed of a cylinder of revolution and two equal spherical segments of one base.

The height of the cylinder $= \sqrt{12^2 - 3^2} = \sqrt{135} = 11.6189$. § 372

The volume of the cylinder $= \pi R^2 H = 3.1416 \times (\tfrac{3}{2})^2 \times 11.6189$ § 700
$= 82.1294$.

The height of each spherical segment $= \frac{1}{2}$ of $(12 - 11.6189)$
$= \frac{1}{2}$ of $0.3811 = 0.1905$.

The volume of each spherical segment $= \pi a^2\left(R - \frac{a}{3}\right)$ § 849

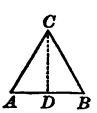
$= 3.1416 \times 0.1905^2 \times \left(6 - \frac{0.1905}{3}\right) = 3.1416 \times 0.1905^2 \times 5.9365$
$= 0.6768$.

Therefore, the volume of the hole $= 82.1294$ cu. in. $+ 2 \times 0.6768$ cu. in.
$= 82.1294$ cu. in. $+ 1.3536$ cu. in.
$= 83.4830$ cu. in.

Therefore, the remaining volume $= 904.7808$ cu. in. $- 83.4830$ cu. in.
$= 821.2978$ cu. in.

Ex. 860. *Find the area of a solid generated by an equilateral triangle turning about one of its sides, if the length of the side is a.*

Let ABC be an equilateral $\triangle$ whose sides are each a.



To find the area of the solid generated by the revolution of the $\triangle ABC$ about AB as an axis.

Let fall the $\perp CD$ from C to the side AB.

Then $BD = \frac{1}{2}a$, § 149

and $CD = \sqrt{\overline{BC}^2 - \overline{BD}^2} = \frac{1}{2}a\sqrt{3}$. § 372

Now it is evident that the area of the solid generated by the revolution of the $\triangle ABC$ about AB is twice the convex surface of the cone generated by the revolution of the rt. $\triangle CBD$ about BD.

Now the convex surface of the cone generated by the revolution of the rt. $\triangle CBD$ about BD is

$$\pi RL = \pi \times CD \times BC = \pi \times \tfrac{1}{2}a\sqrt{3} \times a = \tfrac{1}{2}\pi a^2\sqrt{3}. \quad \text{§ 723}$$

$\therefore$ the required surface $= 2 \times \frac{1}{2}\pi a^2\sqrt{3} = \pi a^2\sqrt{3}$.

Ex. 861. *Compare the volumes of the solids generated by a rectangle turning successively about two adjacent sides, the lengths of these sides being a and b.*

If the rectangle is revolved about the side a, $H = a$, and $R = b$.

$$\therefore V = \pi R^2 H = \pi b^2 a. \quad \text{§ 700}$$

If the rectangle is revolved about the side b, $H = b$, and $R = a$.

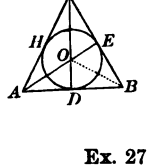
$$\therefore V' = \pi R^2 H = \pi a^2 b. \quad \text{§ 700}$$

$$\therefore \frac{V}{V'} = \frac{\pi b^2 a}{\pi a^2 b} = \frac{b}{a}.$$

Ex. 862. *An equilateral triangle revolves about one of its altitudes. Compare the lateral area of the cone generated by the triangle and the surface of the sphere generated by the inscribed circle.*

Let HDE be the $\odot$ inscribed in the equilateral $\triangle ABC$, each of whose sides is a.

Since the $\triangle$ is equilateral, each of the altitudes is both a median of the $\triangle$ and a bisector of an $\angle$; hence, each of the altitudes CD and AE bisects the $\triangle$ and passes through the centre of the $\odot$. The lines OD and OE are radii drawn to the points of contact of the tangents AB and BC.



$OD = \frac{1}{3} CD.$ Ex. 27

But $CD = \sqrt{CB^2 - \overline{DB}^2} = \frac{1}{2} a\sqrt{3}.$ § 372

Hence, $OD = \frac{1}{6} a\sqrt{3}.$

Convex surface of cone $= \pi RL = \frac{1}{2} \pi a^2.$ § 723

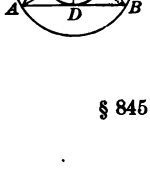
Surface of sphere $= 4\pi R^2 = 4\pi (\frac{1}{6} a\sqrt{3})^2 = \frac{1}{3} \pi a^2.$ § 824

$$\therefore \frac{\text{surface of cone}}{\text{surface of sphere}} = \frac{\frac{1}{2}\pi a^2}{\frac{1}{3}\pi a^2} = \frac{3}{2}.$$

Ex. 863. *An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.*

Let ABC be an equilateral $\triangle$, each of whose sides is a; and let HDE be the inscribed, and ABC the circumscribed $\odot$.

As in Ex. 862, the altitude CD bisects the equilateral $\triangle$ and passes through the centre of both the ⑤, and the radius of the inscribed $\odot$ is $\frac{1}{6} a\sqrt{3}$.



The cone: $V = \frac{1}{3} \pi R^2 H.$ § 725

Here $R = \frac{1}{2} a,$

and $H = \frac{1}{2} a\sqrt{3}.$

$$\therefore V = \tfrac{1}{3} \pi (\tfrac{1}{2} a)^2 \times \tfrac{1}{2} a\sqrt{3}$$
$$= \tfrac{1}{24} \pi a^3 \sqrt{3}.$$

The inscribed sphere: $V_1 = \frac{4}{3} \pi R^3.$ § 845

Here $R = \frac{1}{6} a\sqrt{3}.$

$$\therefore V_1 = \tfrac{4}{3} \pi (\tfrac{1}{6} a\sqrt{3})^3 = \tfrac{1}{54} \pi a^3 \sqrt{3}.$$

The circumscribed sphere:

$V_2 = \frac{4}{3} \pi R^3.$ § 845

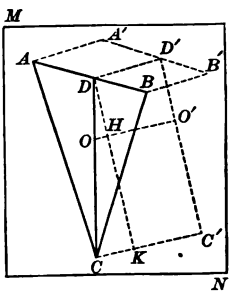
Here $R = OB = \frac{2}{3}$ of $\frac{1}{2} a\sqrt{3} = \frac{1}{3} a\sqrt{3}.$ Ex. 27

$$\therefore V_2 = \tfrac{4}{3} \pi (\tfrac{1}{3} a\sqrt{3})^3 = \tfrac{4}{27} \pi a^3 \sqrt{3}.$$

$$\therefore V : V_1 : V_2 = \tfrac{1}{24} \pi a^3 \sqrt{3} : \tfrac{1}{54} \pi a^3 \sqrt{3} : \tfrac{4}{27} \pi a^3 \sqrt{3}$$
$$= 9 : 4 : 32.$$

Ex. 864. *The perpendicular let fall from the point of intersection of the medians of a given triangle upon any plane not cutting the triangle is equal to one third the sum of the perpendiculars from the vertices of the triangle upon the same plane.*

Let OO' be the $\perp$ from O, the intersection of the medians of the $\triangle ABC$, to the plane MN which does not cut the $\triangle$, and let AA', BB', and CC' be the $\perp$s from the vertices of the $\triangle$ to the same plane.



To prove that

$$OO' = \tfrac{1}{3}(AA' + BB' + CC').$$

Proof. Suppose CC' to be the greatest of these $\perp$s.

Draw $DD' \perp$ to the plane from the middle point D of the side AB, and through D draw, in plane DCC', $DK \parallel$ to the plane MN.

Now AA', DD', and BB' all lie in the same plane, whose intersection with MN is $A'D'B'$.

In the trapezoid $AA'B'B$, $AA' + BB' = 2\,DD'$. § 190

The $\perp$s DD', OO', CC' all lie in the same plane, and

$$2\,DD' + KC' = 3\,HO'. \qquad \text{§ 180}$$

In the similar $\triangle DCK$ and DHO,

$$DC = 3\,DO. \qquad \text{Ex. 27}$$

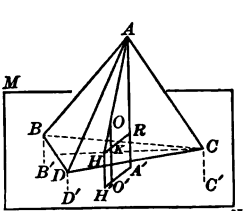
$$\therefore KC = 3\,OH. \qquad \text{§ 351}$$

$$\therefore AA' + BB' + CC' = 2\,DD' + KC' + CK = 3\,(HO' + OH).$$

That is, $$OO' = \tfrac{1}{3}(AA' + BB' + CC'). \qquad \text{Q. E. D.}$$

Ex. 865. *The perpendicular from the centre of gravity of a tetrahedron to a plane not cutting the tetrahedron is equal to one fourth the sum of the perpendiculars from the vertices of the tetrahedron to the plane.*

Let OO' be the $\perp$ from O, the centre of gravity of the tetrahedron $ABCD$, to the plane MN, which does not cut the tetrahedron, and let AA', BB', CC', and DD' be the $\perp$s from the vertices of the tetrahedron to the same plane.



To prove that

$$OO' = \tfrac{1}{4}(AA' + BB' + CC' + DD').$$

Proof. Let H be the intersection of the medians of the face BDC.

Then O lies on the line HA at a distance from H equal to $\frac{1}{4}\,HA$. Ex. 829

Through H, in the plane HAA', draw $HR \parallel$ to the plane MN, cutting OO at K.

Now $BB' + CC' + DD' = 3\,HH'$. Ex. 864

$3\,HH' + RA' = 4\,KO'$. § 180

In the similar $\triangle HAR$ and HOK,

$HA = 4\,HO$. Ex. 829

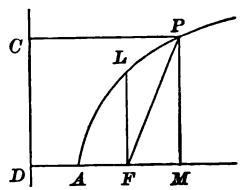
$\therefore AR = 4\,OK$. § 351

Hence, $BB' + CC' + DD' + AA' = 3\,HH' + RA' + AR$

$= 4\,(KO' + OK) = 4\,OO'$.

That is, $OO' = \frac{1}{4}\,(AA' + BB' + CC' + DD')$. Q. E. D.

Ex. 866. *If the abscissa of a point is equal to the ordinate, each is equal to the latus rectum.*



Let AM, the abscissa of the point P, be equal to PM, the ordinate of the point P.

To prove that $AM = PM = 4\,AF$.

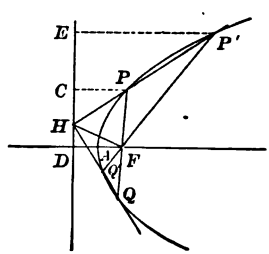
Proof. Now $\overline{PM}^2 = 4\,AF \times AM$. § 862

But $PM = AM$. Hyp.

$\therefore PM = 4\,AF$.

$\therefore AM = PM = 4\,AF$. Q. E. D.

Ex. 867. *If a secant PP' meets the directrix at H, then HF is the bisector of the exterior angle between the focal radii FP and FP'.*



Let the secant PP' cut the directrix at H, and let PF and $P'F$ be the focal radii of the points P and P'.

To prove that HF bisects the exterior $\angle$ of PFP'.

Proof. Draw PC and $P'E \perp$ to the directrix.

Then the rt. $\triangle HPC$ and $HP'E$ are similar. § 356

$\therefore PH : P'H = PC : P'E$. § 351

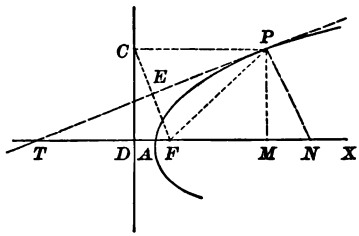
But $PC = PF$, and $P'E = P'F$. § 850

$\therefore PH : P'H = PF : P'F$.

$\therefore HF$ is the exterior bisector of the $\angle PFP'$. § 349

Q. E. D.

Ex. 868. *To draw a tangent and a normal at a given point of a parabola.*



Let P be a given point of the parabola PA.

To draw the tangent and the normal at P.

Construction. (1) Draw the ordinate PM.

On the axis lay off AT equal to AM, and draw PT.

Then PT is the tangent required. § 876

Draw $PN \perp$ to PT at P.
Then PN is the normal required. § 874

(2) Lay off MN equal to $2\,AF$, and draw PN.
Then PN is the normal required § 877
Draw $PT \perp$ to PN at P.
Then PT is the tangent required. § 874

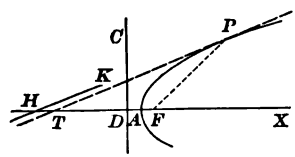
(3) Draw FP, and draw $PC \perp$ to the directrix.
Draw PT bisecting the $\angle FPC$.
Then PT is the tangent required. § 869

Q. E. F.

Ex. 869. *To draw a tangent to a parabola parallel to a given line.*

Let F be the focus of the given parabola, and KH the given line.

To draw a tangent to the parabola ∥ to KH.



Construction. Draw the focal radius FP, making the $\angle XFP$ equal to $2\angle DHK$.
Draw PT the tangent at P. Ex. 868
Then PT is the tangent required.

Proof. $\angle XFP = \angle FTP + \angle FPT = 2\angle FTP$. §§ 137, 871
But $\angle XFP = 2\angle DHK$. Const.
$\therefore \angle FTP = \angle DHK$. Ax. 7
$\therefore PT$ is ∥ to KH. § 114

Q. E. F.

Ex. 870. *The tangents at the ends of the latus rectum meet at D.*

To prove that the tangents at the ends of the latus rectum meet at D.

Proof. $FP = FT$. § 871

Now if P is at the end of the latus rectum,

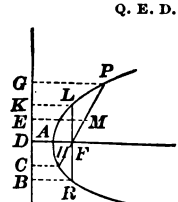
$FP = 2\,AF$. § 861
$\therefore FT = 2\,AF = FD$.

Therefore, the tangents at the ends of the latus rectum meet at D.

Q. E. D.

Ex. 871. *The latus rectum is the shortest focal chord.*

Let LR be the latus rectum and PFH any other focal chord.

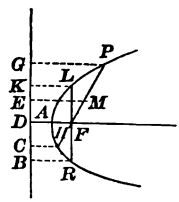


To prove that $LR < PH$.

Proof. Draw PG, LK, HC, and $RB \perp$ to the directrix.

Since $GP > CH$, $FP > FH$. § 850

Hence, M, the middle point of PH, lies in FP, and therefore to the right of LR.



Draw $ME \perp$ to the directrix.

Then $2\,DF < 2\,EM$.

But $LR = 2\,DF$, § 861

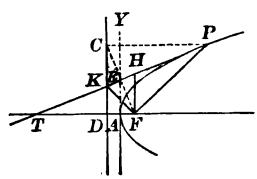
and $PH = FP + FH = GP + CH = 2\,EM$.

§§ 850, 190

$\therefore LR < PH$. Q. E. D.

Ex. 872. *The tangent at any point meets the directrix and the latus rectum produced at points equally distant from the focus.*

Let PT, the tangent at any point P, meet the directrix at K and the latus rectum produced at H.



To prove that $FK = FH$.

Proof. Let E be the foot of the $\perp$ from F to PT.

Then E lies in AY, the tangent at A. § 873

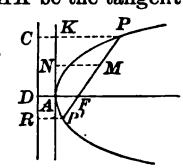
Since $AD = AF$, then $EK = EH$. § 187

$\therefore FK = FH$. § 160

Q. E. D.

Ex. 873. *The circle whose diameter is FP touches the tangent at A.*

Let FP be the focal radius of any point P, and let AK be the tangent at A.



To prove that the $\odot$ with diameter FP is tangent to AK.

Proof. Draw $PC \perp$ the directrix, cutting AK at K, and from M, the middle point of FP, draw $MN \perp$ to AK.

AK is $\parallel$ to the directrix. § 872

$FP = PC = PK + KC = PK + AF = 2\,NM$. §§ 850, 190

$\therefore FM = NM$.

Hence, the $\odot$ whose centre is M and radius MF is tangent to AK. § 253

Q. E. D.

Ex. 874. *The directrix touches the circle that has any focal chord for diameter.*

Let PP' be any focal chord.

To prove that the $\odot$ with diameter PP' is tangent to the directrix.

Proof. Draw PC and $P'R \perp$ to the directrix.

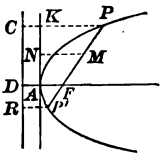
Then $PP' = PF + P'F = PC + P'R$. § 850

$\therefore \frac{1}{2} PP' = \frac{1}{2}(PC + P'R)$. Ax. 7

That is, $\frac{1}{2} PP'$ equals the distance from the middle point of PP' to the directrix. § 190

Hence, the ⊙ on PP' as diameter is tangent to the directrix. § 253

Q. E. D.



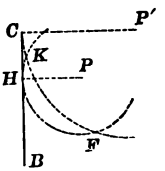
Ex. 875. *Given two points and the directrix, to find the focus.*

Let P and P' be two given points of a parabola, and BC the directrix. To find the focus.

Construction. From P and P' let fall the ⊥s PH and $P'C$ upon the directrix BC.

From P and P' as centres, and PH and PC respectively as radii, describe arcs intersecting in F and K.

Then either F or K is the required focus.



Proof. Since P and P' are each equally distant from K and BC, the parabola whose focus is K, and whose directrix is BC, passes through P and P'. §§ 850, 866

For like reason, the parabola whose focus is F, and whose directrix is BC, passes through P and P'. Q. E. F.

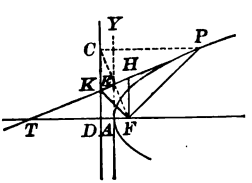
Ex. 876. *The perpendicular FC bisects TP.* (See figure, page 414.)

To prove that FC bisects TP.

Proof.

$FP = FT$.	§ 871
FC is ⊥ to TP.	Hyp.
$\therefore FC$ bisects TP.	§ 149

Q. E. D.



Ex. 877. *Given the focus and the axis, to describe a parabola which shall touch a given straight line.*

Let F be the focus and TX the axis of a parabola, and let PT be the given straight line.

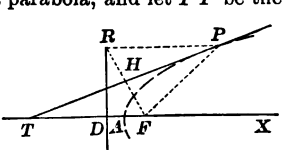
To construct the parabola tangent to PT.

Construction. From F draw FH ⊥ to TP, and produce it to R, making HR equal to FH.

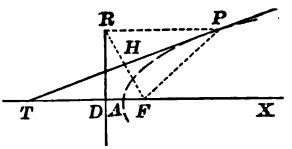
Draw RD ⊥ to TX, and RP ∥ to TX.

Describe the parabola AP, having F as focus and DR as directrix.

Then AP is the parabola required.



Proof. Draw FP to the point of intersection of TP and RP.



Since $HR = HF$, and RF is $\perp$ to TP,

Const.

$PR = PF$. § 160

Hence, P is on the parabola AP. § 850

$\angle FPT = \angle RPT$. § 95

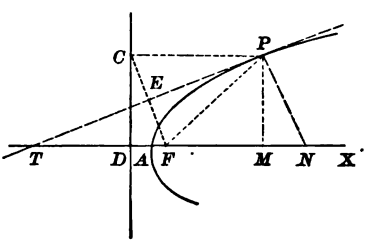
Hence, TP is tangent to AP at P. § 869

Q. E. F.

Ex. 878. *If PN is any normal, and the triangle PNF is equilateral, then PF is equal to the latus rectum.*

Let PN be the normal of the point P, and let the $\triangle PNF$ be equilateral.

To prove that $PF = 4\,AF$.



Proof. Draw the ordinate PM.

PM bisects FN. § 149

$\therefore FM = MN = 2\,AF$. § 877

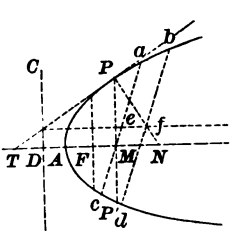
$\therefore PF = FN = 4\,AF$.

Q. E. D.

Ex. 879. *Given a parabola, to find the directrix, axis, and focus.*

Let $P'AP$ be the given parabola.

To find the directrix, axis, and focus.



Construction. Draw any two parallel chords ac and bd, bisect them in e and f, and draw ef.

Then the line ef is a diameter. § 888

From any point P of the parabola draw PP' $\perp$ to ef, and through its middle point M draw $MT \parallel$ to ef, intersecting the parabola at A.

Then MT is the axis of the curve, and A the vertex.

For the axis bisects chords $\perp$ to the diameters.

Take AT equal to AM and draw TP.

Then PT is the tangent at P. § 876

Draw the normal PN.

Then MN is equal to the semi-latus rectum. § 877

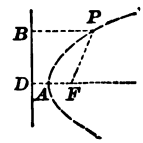
Take AF equal to AD equal to $\frac{1}{4}\,MN$.

Then F is the focus, and the $\perp$ to TM at D is the directrix. Q. E. F.

Ex. 880. *To find the locus of the centre of a circle which passes through a given point and touches a given straight line.*

Let F be the given point and DB the given line.

To find the locus of the centre of the ⊙ which passes through F and touches DB.



Construct the parabola AP with F as focus and DB as directrix.

Then AP is the required locus.

Proof. Let P be any point on this parabola.

Draw FP, and draw $PB \perp$ to DB.

Then, since $PF = PB$, the ⊙ with P as centre and PF as radius passes through F and is tangent to DB. § 253

Now any point within the parabola is nearer to the focus than to the directrix, and any point without is nearer to the directrix than to the focus. § 865

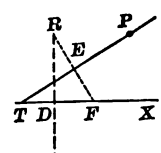
Therefore, the parabola is the locus required. Q. E. F.

Ex. 881. *Given the axis, a tangent, and the point of contact, to find the focus and directrix.*

Let TX be the axis, TP a tangent, and P the point of contact.

To construct the focus and the directrix.

Construction. Bisect TP; at E, the middle point, erect $REF \perp$ to TP, intersecting TX in F.



Then F is the focus.

Lay off ER equal to EF, and draw $RD \perp$ to TX.

Then DR is the directrix.

Proof. The ⊥ which bisects TP passes through the focus. Ex. 876

Hence, F is the focus.

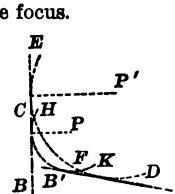
The tangent bisects FR.

Hence, DR is the directrix. Q. E. F.

Ex. 882. *Given two points and the focus, to find the directrix.*

Let P and P' be two points of a parabola, and F the focus.

To construct the directrix.



Construction. With P as centre and PF as radius, draw the arc KH.

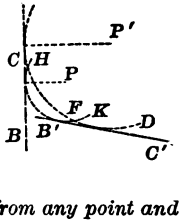
With P' as centre and $P'F$ as radius, draw the arc DE.

Draw the common tangents BC and $B'C'$.

Then either BC or $B'C'$ is the directrix required.

Proof. Since P and P' are each, by construction and § 254, equally distant from F and BC, the parabola whose focus is F and whose directrix is BC, passes through both P and P'. §§ 850, 866

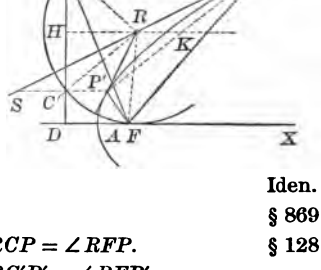
For like reason, the parabola whose focus is F and whose directrix is $B'C'$ passes through both P and P'. Q. E. F.



Ex. 883. *The triangles formed by the two tangents from any point and the focal radii to the points of contact are similar.*

Let RP and RP' be the tangents from R, and let PF and $P'F$ be the focal radii of P and P'.

To prove that the $\triangle RPF$, and $P'RF$ are similar.



Proof. Draw PC and $P'C' \perp$ to the directrix.

Draw RC and RC'.

The $\triangle FPR$ and CPR are equal. § 143

For $FP = CP$, § 850

$PR = PR$, Iden.

and $\angle FPR = \angle CPR$. § 869

$\therefore FR = RC$, and $\angle RCP = \angle RFP$. § 128

In like manner, $C'R = FR$, and $\angle RC'P' = \angle RFP'$.

$\therefore FR = RC = RC'$. Ax. 1

With R as centre and RC as radius, describe a circle through C, C', F.

Now $\angle FPR = \angle RPC$. § 869

$\angle RPC = \angle FCC'$. Ex. 90

$\angle FCC' = \frac{1}{2}\angle FRC'$.

For $\frac{1}{2}$ the arc $C'F$ is the measure of the $\angle FCC'$, § 289

and the arc $C'F$ is the measure of the $\angle FRC'$. § 288

But $\angle FRP' = \angle P'RC'$. § 128

$\therefore \angle FRP' = \angle FCC' = \angle FPR$.

Again $\angle RFP = \angle RFP'$. § 883

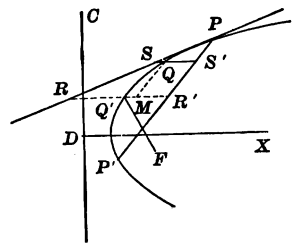
$\therefore$ the $\triangle RPF$ and $P'RF$ are similar. § 355

Q. E. D.

Ex. 884. *If a diameter of a parabola is cut by a chord and the tangent at either end of the chord, the segments of the diameter between the tangent*

and the chord made by the curve are in the same ratio as the segments of the chord.

Let the diameter SQS' meet the chord PP' at S' and the tangent PR at S.



To prove that $SQ : QS' = PS' : S'P'$.

Proof. Draw RQR' the diameter of which PP' is an ordinate; draw $QM \parallel$ to PP' meeting RR' at M, and draw $Q'F$.

Now $\overline{PR'}^2 = 4\,\overline{FQ} \times Q'R'$,

and $\overline{QM}^2 = 4\,FQ \times Q'M$. § 862

$\therefore \overline{PR'}^2 - \overline{QM}^2 = 4\,FQ \times MR'$. Ax. 3

But $\overline{PR'}^2 - \overline{QM}^2 = \overline{PR'}^2 - \overline{Q'R'}^2$

$= PS' \times SP'$,

and $MR' = QS'$.

$\therefore PS' : S'P' = 4\,FQ' : QS'$.

$\therefore QS' : S'P' = PS' : 4\,FQ'$.

In the similar $\triangle PS'S$ and $PR'R$,

$SS' : PS' = RR' : PR'$ § 351

$= 2\,Q'R' : PR'$ § 885

$= PR' : 2\,Q'F$

$= PP' : 4\,Q'F$.

$\therefore SS : PP' = PS' : 4\,Q'F$.

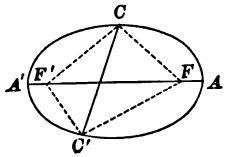
$\therefore QS' : S'P' = SS' : PP'$.

$\therefore QS' : SS' = S'P' : PP'$.

$\therefore SQ : QS' = PS' : S'P'$. Q. E. D.

Ex. 885. *The major axis is the longest chord that can be drawn in an ellipse.*

Let AA' be the major axis of the ellipse $ACA'C'$ and CC' any other chord.



To prove that $AA' > CC'$.

Proof. Draw the focal radii CF, CF', $C'F$, $C'F'$.

$CC' < F'C + F'C'$, § 138

and $CC' < FC + FC'$.

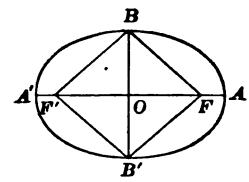
$\therefore 2\,CC' < (F'C + FC) + (F'C' + FC') < 4a$. Ax. 4

$\therefore CC' < 2a$, or AA'. Ax. 7

Q. E. D.

Ex. 886. *If the angle FBF' is a right angle, then $a^2 = 2b^2$.*

In the ellipse $ABA'B'$, let the $\angle FBF'$ be a rt. $\angle$.



To prove that $a^2 = 2b^2$.

Proof. $\overline{F'B}^2 + \overline{FB}^2 = \overline{FF'}^2$. § 371

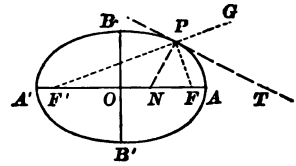
That is, $2a^2 = (2c)^2 = 4c^2 = 4(a^2 - b^2)$.

$\therefore 2a^2 = 4b^2$.

$\therefore a^2 = 2b^2$. Q. E. D.

Ex. 887. *To draw a tangent and a normal at a given point of an ellipse.*

Let P be any given point of the ellipse $ABA'B'$.



To draw the tangent and the normal at P.

Construction. Draw the focal radii FP and $F'P$.

Produce $F'P$ to some point G.

Draw PT bisecting the $\angle FPG$.

Then PT is the tangent required. § 925

At P draw $PN \perp$ to PT.

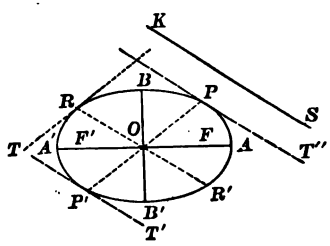
Then PN is the normal at P. § 933

Q. E. F.

Ex. 888. *To draw a tangent to an ellipse parallel to a given straight line.*

Let $ABA'B'$ be the given ellipse and KS the given line.

To draw a tangent to the ellipse $\parallel$ to KS.



Construction. Draw the diameter $RR' \parallel$ to KS, and at R draw the tangent RT. Ex. 887

Draw the diameter $PP' \parallel$ to RT.

At P and P' draw the tangents PT'' and $P'T'$.

PT'' and $P'T'$ are the tangents required.

Proof. The diameter PP' is conjugate to RR'. Const.

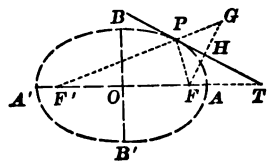
$\therefore PT''$ and $P'T'$ are $\parallel$ to RR' or its $\parallel KS$. § 939

Q. E. F.

Ex. 889. *Given the foci; to describe an ellipse touching a given straight line.*

Let F and F' be the given foci, and PT the given line.

To construct the ellipse tangent to PT, having F and F' for foci.



Construction. Draw $FH \perp$ to PT, and produce it to G, making HG equal to FH.

Draw $F'G$, cutting the tangent in some point P. Draw PF.

Construct the ellipse $AA'P$, with F and F' as foci, and $F'G$ or $F'P + PF$ as the constant sum.

Then $AA'P$ is the ellipse required.

Proof. F and F' are its foci, and it passes through P. Const.

PH bisects FG at right angles. Const.

Hence, PH bisects $\angle FPG$, and is therefore a tangent at P. § 925

Q. E. F.

Ex. 890. *Prove that* $\overline{OF}^2 = OT \times ON$. (See figure, page 436.)

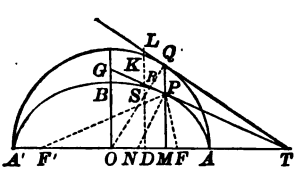
To prove that $\overline{OF}^2 = OT \times ON$.

Proof. $OT = \frac{a^2}{d}$. § 936

$ON = e^2 d$. § 936

$\therefore OT \times ON = \frac{a^2}{d} \times e^2 d = a^2 e^2 = \overline{OF}^2$.

Q. E. D.



Ex. 891. *Prove that* $OM : ON = a^2 : c^2$. (See figure, page 436.)

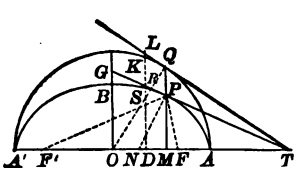
To prove that $OM : ON = a^2 : c^2$.

Proof. $OM = d$. § 936

$ON = e^2 d$. § 936

$\therefore OM : ON = d : e^2 d = 1 : e^2$

$= a^2 : a^2 e^2 = a^2 : c^2$. Q. E. D.

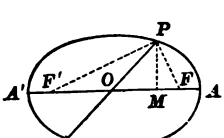


Ex. 892. *The minor axis is the shortest diameter of an ellipse.*

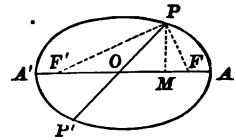
Let PP' be any diameter of the ellipse AA'.

To prove that the minor axis is the shortest diameter.

Proof. Draw the focal radii FP and $F'P$.



$$\overline{OP}^2 + \overline{OF}^2 = \tfrac{1}{2}(\overline{F'P}^2 + \overline{FP}^2) \qquad \text{§ 377}$$
$$= \tfrac{1}{2}(a + ed)^2 + \tfrac{1}{2}(a - ed)^2 \qquad \text{§ 915}$$
$$= a^2 + e^2d^2.$$
$$\therefore OP = \sqrt{a^2 + e^2d^2 - c^2}.$$

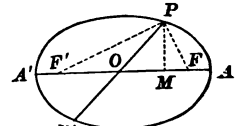


Hence, OP is least when $\sqrt{a^2 + e^2d^2 - c^2}$ is least, that is when $d = 0$.

Hence, OP is least when it coincides with the minor axis. Q. E. D.

Ex. 893. *At what points of an ellipse will the normal pass through the centre of the ellipse?*

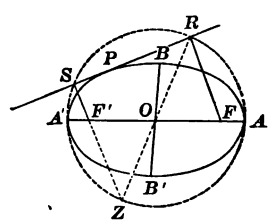
The normal at P, which is the bisector of the $\angle F'PF$, passes through O, the middle point of $F'F$, only when $F'PF$ is an isosceles triangle, or when $\angle F'PF = 0$.



Hence, the normals at the points B, B', A, A', and at these points only, pass through O.

Ex. 894. *If FR, $F'S$ are the perpendiculars dropped from the foci to any tangent, then $FR \times F'S = b^2$.*

Let FR and $F'S$ be the ⊥s dropped from the foci on RS, which is the tangent at P to the ellipse $ABA'B'$.



To prove that $FR \times F'S = b^2$.

Proof. Describe the auxiliary ⊙ of the ellipse.

Produce SF' to meet the auxiliary ⊙ in Z, and draw RZ.

Now R and S lie in the circumference of the auxiliary ⊙. § 928

ZSR is a right $\angle$. Hyp.

$\therefore$ RZ passes through the centre O, and is a diameter of the auxiliary ⊙. § 290

The ▲ ORF and OZF' are equal. § 143

For $OR = OZ$, § 217

$OF = OF'$, § 905

and $\angle FOR = F'OZ$. § 93

$\therefore FR = F'Z$. § 128

$$\therefore FR \times F'S = F'Z \times F'S$$
$$= F'A' \times F'A \qquad \text{§ 378}$$
$$= (a - c)(a + c)$$
$$= a^2 - c^2$$
$$= b^2. \qquad \text{§ 907}$$

Q. E. D.

Ex. 895. *The semi-minor axis of an ellipse is the mean proportional between the segments of the major axis made by one of the foci.*

Let AA' be the major axis of an ellipse, F and F' the foci.

To prove that $F'A' \times F'A = b^2$.

Proof. $F'A' = a - c$, and $F'A = a + c$.

$\therefore F'A' \times F'A = (a - c)(a + c)$

$= a^2 - c^2 = b^2$. § 907

Q. E. D.

Ex. 896. *The area of an ellipse is to the area of its auxiliary circle as the minor axis is to the major axis.*

Let $2b$ be the minor axis and $2a$ the major axis of an ellipse.

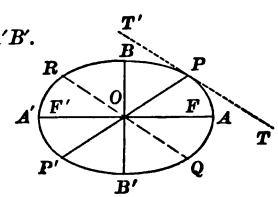
To prove that the ellipse : the auxiliary $\odot = 2b : 2a$.

Proof. The area of the ellipse $= \pi ab$. § 942

The area of the auxiliary $\odot = \pi a^2$. § 463

$\therefore$ ellipse : auxiliary $\odot = \pi ab : \pi a^2 = b : a = 2b : 2a$. Q. E. D.

Ex. 897. *To draw a diameter conjugate to a given diameter in a given ellipse.*



Let PP' be a diameter of the ellipse $ABA'B'$.

To draw a diameter conjugate to PP'.

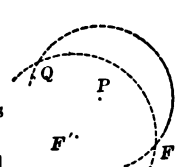
Construction. Draw TT' the tangent at P.

Draw the diameter $QOR \parallel$ to TT'.

Then QOR is the conjugate diameter required. § 938

Q. E. F.

Ex. 898. *Given $2a$, $2b$, one focus, and one point of an ellipse, to construct the ellipse.*



Let F' be one focus and P a point of an ellipse.

Let $2a$ and $2b$ also be given.

To construct the ellipse.

Construction. With P as centre and $2a - PF'$ as radius, describe a circle.

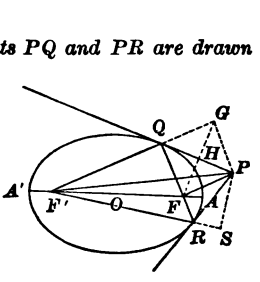
Construct a right triangle with a as hypotenuse and b as one side; then the other side is c. § 907

With F' as centre and $2c$ as radius, describe a circle, intersecting the first in F and Q.

Then the ellipse with F' and F as foci, or F' and Q as foci, and $2a$ as major axis, will be the ellipse required.

Proof. For either will have the focus F' and will pass through P, since $PF' + PF = 2a$, or $PF' + PQ = 2a$, and the axes of each will be $2a$ and $2b$. Q. E. F.

Ex. 899. *If from a point P a pair of tangents PQ and PR are drawn to an ellipse, then PQ and PR subtend equal angles at either focus.*



Let PQ and PR be the tangents from the given point P to the given ellipse $AQA'R$.

To prove that $\angle PF'Q = \angle PF'R$,

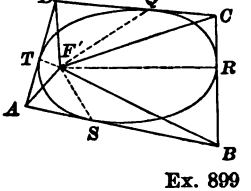
and $\angle PFQ = \angle PFR$.

Proof. Let the tangents be drawn by the method of § 929.

	The $\triangle PF'G$ and $PF'S$ are equal.	§ 150
For	$F'G = F'S = 2a$.	Const.
	$PG = PS$,	Const.
and	$PF' = PF'$.	Iden.
	$\therefore \angle PF'Q = PF'R$.	§ 128
	The $\triangle PGQ$ and PFQ are equal.	§ 150
For	$FG = FP$,	Const.
	$QG = QF$,	Const.
and	$PQ = PQ$.	Iden.
	$\therefore \angle PFQ = \angle PGQ$.	§ 128
In like manner,	$\angle PFR = \angle PSR$.	
But	$\angle PGQ = \angle PSR$.	§ 128
	$\therefore \angle PFR = \angle PFQ$.	Ax. 1
		Q. E. D.

Ex. 900. *If a quadrilateral is circumscribed about an ellipse, either pair of its opposite sides subtend angles at either focus whose sum is equal to two right angles.*

Let $ABCD$ be a quadrilateral circumscribed about the ellipse $RQTS$. Let the vertices A, B, C, D be joined to F', one of the foci.



Ex. 899

To prove that

$\angle AF'D + \angle CF'B = 2$ rt. $\measuredangle$,

and $\angle AF'B + \angle CF'D = 2$ rt. $\measuredangle$.

Proof. Draw to the points of tangency the focal radii $F'Q$, $F'R$, $F'S$, $F'T$.

Now $\angle AF'T = \angle SF'A$,

$\angle TF'D = \angle DF'Q$,

$\angle CF'R = \angle CF'Q$,

and $\angle RF'B = \angle BF'S$.

$\therefore \angle AF'T + \angle TF'D + \angle CF'R + \angle RF'B$
$= \angle SF'A + \angle DF'Q + \angle CF'Q + \angle BF'S$. Ax. 2

That is, $\angle AF'D + \angle CF'B = \angle AF'B + \angle CF'D$.

But the sum of these four $\measuredangle = 4$ rt. $\measuredangle$. § 73

$\therefore \angle AF'D + \angle CF'B = 2$ rt. $\measuredangle = \angle AF'B + \angle CF'D$. Q. E. D.

Ex. 901. *To find the foci of an ellipse, having given the major axis and one point on the curve.*

Let AA' be the major axis of an ellipse, and P a point on the curve.

To find the foci of the ellipse.

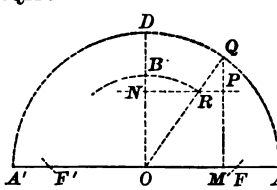
Construction. Draw the auxiliary circle AQA'.

At O, the middle point of AA', erect $OD \perp$ to AA'.

Draw $PM \perp$ to AA', and produce it to meet the auxiliary circle in Q.

Draw OQ, and draw $PN \parallel$ to AA' cutting OQ in R.

On OD lay off OB equal to OR.



With B as centre and OA as radius, strike arcs, cutting AA' in F and F'.

Then F and F' are the foci required.

Proof. P is a point on the ellipse whose major axis is AA', whose foci are F and F', and whose semi-minor axis is OB. § 918

Q. E. F.

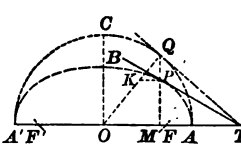
Ex. 902. *To find the foci of an ellipse, having given the major axis and a straight line which touches the curve.*

Let AA' be the major axis of an ellipse and PT a tangent.

To find the foci.

Construction. Draw the auxiliary circle ACA', and from T, the intersection of PT and AA' produced, draw the tangent TQ.

Draw QO, and draw $QM \perp$ to AA', intersecting PT in P.



Through P draw $PK \parallel$ to AA', intersecting QO in K.

On OC, the $\perp$ to AA' at O, lay off OB equal to OK, and from B as centre, with OA as radius, draw arcs cutting $A'A$ in F' and F.

Then F' and F are the foci required.

Proof. MT is the subtangent. § 931

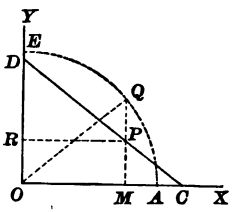
Hence, P is the point of contact.

$OK = b$. § 918

Since $BF' = BF = a$, F' and F are the foci required. Q. E. F.

Ex. 903. *If a straight line moves so that its extremities are always in contact with two fixed straight lines perpendicular to each other, then any point of the moving line describes an ellipse.*

Let the straight line DC move so that its extremities C and D are always in the perpendicular lines OX and OY, respectively.



To prove that any point P in this line describes an ellipse.

Proof. With O as centre, and DP, the longer segment, as radius, draw the arc AQE.

Draw $PM \perp$ to OX, and $PR \parallel$ to OX.

Produce MP, meeting the circle in Q, and draw OQ.

The rt. $\triangle OMQ$ and RPD are equal. § 151

For $OM = RP$, § 180

and $QO = DP$. Const.

$\therefore RD = MQ$. § 128

The rt. $\triangle PMC$ and DRP are similar. § 356

$\therefore MP : RD = CP : PD$. § 351

$\therefore MP : MQ = CP : OQ$.

Hence, as Q is in the circle whose radius is OQ, P is in the ellipse whose semi-axes are OQ and PC, or DP and PC. § 917

Q. E. D.

Ex. 904. *To construct an ellipse, having given one of the foci and three tangents.*

Let F' be a focus of an ellipse, and ED, BC, HI be three given tangents.

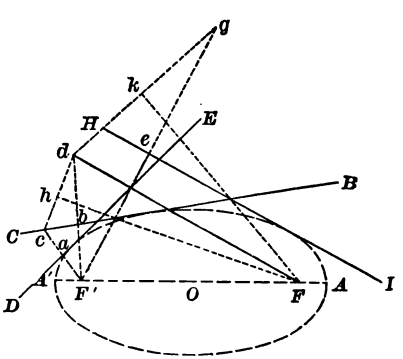
To construct the ellipse.

Construction. Draw $F'a \perp$ to DE, and produce it to c, making ac equal to $F'a$.

Draw $F'b \perp$ to BC, and produce it to d, making bd equal to $F'b$.

Draw $F'e \perp$ to HI, and produce it to g, making eg equal to $F'e$.

Draw cd and dg, and bisect them with the $\perp$s hF and kF, respectively.



These ⊾ intersect in some point F.

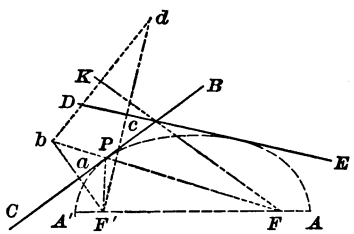
Construct the ellipse, having F' and F as foci, and dF as the major axis.

This ellipse is the ellipse required.

Proof. By proof in Ex. 889, the ellipse having F' and F as foci, and cF equal to dF equal to gF as major axis, is tangent to each of the lines, DE, CB, HI. Q.E.F.

Ex. 905. *To construct an ellipse, having given one focus, two tangents, and one of the points of contact.*

Let F' be a focus of an ellipse, BC and DE be tangents, and P be the point of tangency of BC.



To construct the ellipse.

Construction. Draw $F'P$, and draw PF, making the $\angle BPF$ equal to the $\angle CPF'$.

Draw $F'a \perp$ to BC and produce it to b, making ab equal to $F'a$.

Draw $F'c \perp$ to DE and produce it to d, making cd equal to $F'c$.

Draw bd and bisect it with the $\perp KF$, intersecting PF in F.

Construct the ellipse, having F and F' as foci and bF, or $F'P + PF$, as major axis.

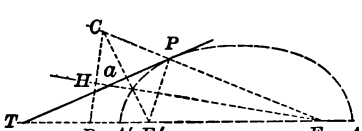
This ellipse is the ellipse required.

Proof. The ellipse having F' and F as foci, and bF or dF as major axis, will be tangent to DE, and tangent to BC at P, by proof in Ex. 889. Q.E.F.

Ex. 906. *To construct an ellipse, having given one focus, one vertex, and one tangent.*

Let F' be a focus, A' a vertex of an ellipse, and PT a tangent.

To construct the ellipse.



Construction. Through A' and F' draw the line AT.

Draw $F'a \perp$ to TP and produce it to C, making aC equal to $F'a$.

Lay off $A'D$ equal to $A'F'$.

Draw DC, and bisect it with the $\perp HF$, intersecting AA' in F.

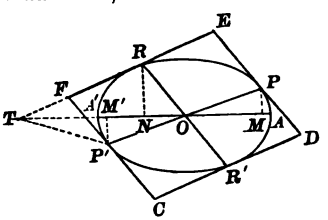
Lay off FA equal to $F'A'$, and construct the ellipse having F' and F as foci and $A'A$ as the major axis.

This ellipse is the ellipse required.

Proof. This ellipse will have F' as a focus, A' as a vertex, and by proof in Ex. 889 will be tangent to TP at P. Q.E.F.

Ex. 907. *The area of the parallelogram formed by the tangents to an ellipse at the extremities of any pair of conjugate diameters is equal to the area of the rectangle contained by the axes of the ellipse.*

Let PP' and RR' be a pair of conjugate diameters, and let the $\square\ CDEF$ be formed by the tangents at P, P', R, R'.



To prove that $\square\ CDEF = 4\,ab$.

Proof. Let the tangent at R meet OA' produced in T.

Draw $P'T$, and draw the ordinates PM, RN, and $P'M'$.

Since RT is $\parallel$ to PO,

$$\square\ P'ORF = 2 \times \triangle\ P'OT = OT \times M'P'. \qquad \text{§§ 400, 403}$$

Since $MP = M'P'$, $M'P' : ON = b : a$. § 941

$$\therefore OT \times M'P' : OT \times ON = b : a = ab : a^2. \qquad \text{§ 340}$$

But $OT \times ON = a^2$. § 932

$$\therefore OT \times M'P' = ab.$$

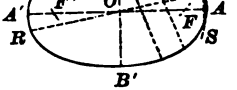
$$\therefore \square\ P'ORF = ab.$$

But $\square\ CDEF = 4 \square\ P'ORF$.

$$\therefore \square\ CDEF = 4\,ab. \qquad \text{Q. E. D.}$$

Ex. 908. *Given an ellipse, to find by construction the centre, the foci, and the axes.*

Let $ABA'B'$ be a given ellipse.



To find by construction the centre, the foci, and the axes.

Construction. Draw any two $\parallel$ chords, and through their middle points draw KOR.

The point O, the middle point of the diameter, is the centre required.

With O as centre and OK as radius, describe an arc, intersecting the ellipse in the points K and S.

Draw the chord KS, which is a chord of one of the axes, since its extremities K and S are equidistant from the centre O.

The two diameters, one $\perp$ to KS, and the other $\parallel$ to KS, are the axes required; the longer AA' is the major axis, and the shorter BB' the minor axis.

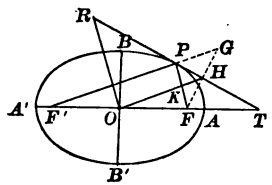
With B as centre and OA as radius, describe arcs, intersecting AA' in F and F'.

F and F' are the foci required.

Proof. The proof is obvious from the construction. Q. E. F.

Ex. 909. *The circle described on any focal radius of an ellipse as a diameter is tangent to the auxiliary circle.*

Let P be any point on the ellipse $ABA'B'$, and FP and $F'P$ be its focal radii.



To prove that the ⊙ described on FP as a diameter is tangent to the auxiliary ⊙.

Proof. Produce $F'P$ to G, making PG equal to PF, and draw FG.

Through O draw $OH \parallel$ to $F'G$, intersecting FP and FG in K and H, respectively.

OH bisects FP and FG, § 188

and, therefore, passes through the centre of the ⊙ on FP as diameter. § 217

Now $OH = \frac{1}{2} F'G = a.$ § 189

$\angle KHP = \angle F'PR.$ § 112

$\angle F'PR = \angle HPG.$ § 93

$\angle HPG = \angle KPH.$ § 145

$\angle KHP = \angle KPH.$ Ax. 1

$\therefore KP = KH.$ § 147

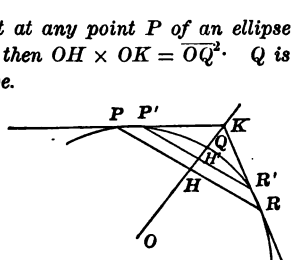
$OK = OH - KH = a - KP.$

Hence, the ⊙ on FP as diameter is tangent to the auxiliary ⊙, since the distance between their centres is equal to the difference of their radii.

Q. E. D.

Ex. 910. *If the ordinate and the tangent at any point P of an ellipse meet a diameter at H and K, respectively, then $OH \times OK = \overline{OQ}^2$. Q is the point in which the diameter cuts the curve.*

Let the ordinate and the tangent at P meet a diameter OQ in H and K, respectively.



To prove that $OH \times OK = \overline{OQ}^2$.

Proof. Draw $P'R' \parallel$ to PR and bisected by OQ at H'.

Let PP' produced meet OQ produced at K.

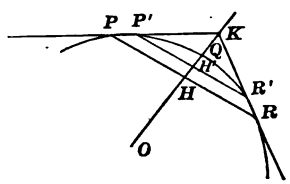
Then $HK : H'K = PH : P'H' = RH : R'H'.$ § 351

$\therefore RR'$ produced also meets OQ produced at K.

Hence, if the $\parallel$ chords are moved up to each other PK and RK become tangents at P and R.

Therefore, the tangents at the extremities of a chord intersect on the diameter to which the chord is conjugate. Also the line that joins the

middle point of a chord to the intersection of the tangents at its extremities passes through the centre of the ellipse.



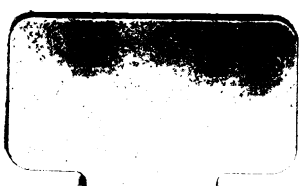
Draw QE the tangent at Q meeting PK in E, and draw $QG \parallel$ to PK meeting PR at G.

Then $PEQG$ is a $\square$ and EG bisects PQ.

$\therefore EG$ passes through the centre O.

$\therefore OH : OQ = OG : OE.$

$\therefore OH : OQ = OQ : OK.$

$\therefore OH \times OK = \overline{OQ}^2.$ Q. E. D.

NOTE. This proof is based on that of the Rev. C. Taylor of St. John's College, Cambridge, Eng.